The Image of Johnson Homormorphism for Compact Surfaces

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Abstract

For any compact surface $\Sigma_{g,b}$ of genus g with b boundary components($g \geq 2, b \geq 1$), there is a representation for its Torelli group $\mathscr{T}(\Sigma_{g,b})$, named the Johnson homomorphism $\tau : \mathscr{T}(\Sigma_{g,b}) \to Hom(H_1(\Sigma_{g,b};\mathbb{Z}), \wedge^2 H_1(\Sigma_{g,b};\mathbb{Z}))$. We compute out the images of Johnson homomorphism for all compact surfaces $\Sigma_{g,b}$ using the matrix form. Especially for $\Sigma_{g,2}$, the image is some form of $\wedge^3 H_1(\Sigma_{g,2};\mathbb{Z})$. In this process, we mainly use the fact that its Torelli group $\mathscr{T}(\Sigma_{g,b})$ is generated by Dehn twists about separating curves and bounding pair maps. Besides, we calculate the image of Johnson homomorphism for one special noncompact surface of infinite type.

1 Introduction

Let $\Sigma_{g,b}$ be a connected, oriented surface of genus g with b boundary components $(g \geq 2, b \geq 1)$; the mapping class group $\mathrm{Mod}(\Sigma_{g,b})$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,b}$ which fix the boundary pointwise. $\mathrm{Mod}(\Sigma_{g,b})$ has a natural representation by acting on its first homology group with $\mathbb{Z}$ coefficients: $\Psi : \mathrm{Mod}(\Sigma_{g,b}) \to \mathrm{Aut}(H_1(\Sigma_{g,b};\mathbb{Z}))$. For $b = 1$, since the action of $\mathrm{Mod}(\Sigma_{g,b})$ on $H_1(\Sigma_{g,b};\mathbb{Z})$ preserves the nondegenerate algebraic intersection form, this representation becomes $\Psi : \mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Sp}(2g,\mathbb{Z})$, which is well-known to be surjective. Its kernel is the Torelli group for $\Sigma_{g,1}$ denoted as $\mathscr{T}(\Sigma_{g,1})$. In this article, we generalized the definition of the Torelli group from $b = 1$ to every postive integer b in a natural way. The Torelli group for $\Sigma_{g,b}(g \geq 2, b \geq 1)$ is defined as the kernel of Ψ. Notice that this definition of Torelli group is different from the definition of the Torelli group given by Putman in [9]. In Putman's article [9], he proved that there is no single definition of the Torelli group that is both funtorial and closed under restriction, so he actually defined the Torelli group $\mathscr{T}(\Sigma, P)$ for a partitioned surface (Σ, P), where P is a partition of the boundary components that restricts how Σ embeds into a closed surface. Although his definition is funtorial, elements in $\mathscr{T}(\Sigma, P)$ do not act trivially on $H_1(\Sigma_{g,b};\mathbb{Z})$, but act trivially on $H_1^P(\Sigma)$ constructed in section 3 of [9]. One important question concerning the Torelli group is finding its generators. Birman and Powell proved that $\mathscr{T}(\Sigma_{g,0})$ is generated by an infinite collection of all Dehn twists about separating twists and all bounding pair maps [1],[8]; Johnson showed that, for $g \geq 3$, a finite number of bounding pair maps can generate $\mathscr{T}(\Sigma_{g,0})$ [5]. For genaral compact surfaces, Putman proved in [9] that $\mathscr{T}(\Sigma, P)$ is generated by Dehn twists about P-separating curves and P-bounding pair maps, while it does not help to get the generators of $\mathscr{T}(\Sigma_{g,b})$ defined in this article. Still, we can show that $\mathscr{T}(\Sigma_{g,b})$ is generated by all Dehn twists about separating curves and bounding pair maps by induction on the genus g, and we give a specific proof due to Justin Malestein in section 2.

The Johnson homomorphism is an important contribution of Johnson to our understanding of the Torelli group. Basically, it's a homomorphism from the Torelli group to a finitely generated abelian group, which is of the form $\tau : \mathscr{T}(\Sigma_{g,1}) \to \mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}), \wedge^2 H_1(\Sigma_{g,1};\mathbb{Z}))$. The original definition for $\Sigma_{g,1}$ first came from Johnson [4]. The problem of interest is the computation of the abelianization of $\mathscr{T}(\Sigma_{g,1})$. In [4], he aimed to construct an abelian quotient of $\mathscr{T}(\Sigma_{g,1})$ free of rank $\binom{3}{2g}$, by examing its action on a certain nilpotent quetient of $\pi_1(\Sigma_{g,a})$, which is given by τ. He showed that $\wedge^3 H_1(\Sigma_{g,1})$ has a natural way of embedding in $\mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}), \wedge^2 H_1(\Sigma_{g,1};\mathbb{Z}))$ and proved that the image of τ is precisely $\wedge^3 H_1(\Sigma_{g,1})$. Besides, he applied Johnson homomorphism to two problems from Birman and Chillingworth. One is the subgroup of $\mathscr{T}(\Sigma_{g,1})$ which is generated by Dehn twists on bounding simple closed curves does not have finite index in $\mathscr{T}(\Sigma_{g,1})$. The other is if an element f in $\mathscr{T}(\Sigma_{g,1})$ preserves winding numbers of all curves on $\Sigma_{g,1}$, it's not necessary that f belongs to the subgroup of $\mathscr{T}(\Sigma_{g,1})$ which is generated by Dehn twists on bounding simple closed curves. Returning to the original problem about the abelianization of $\mathscr{T}(\Sigma_{g,1})$, Johnson also showed that for $g \geq 3$, $\wedge^3 H_1(\Sigma_{g,1})$ can't be the abelianization. For surfaces with more boundary components, we can similarly define the Johnson homomorphism as [4], and what's its image like is of our concern.

The main purpose of this paper is computing the image of the Johnson homomorphism for general compact surfaces $\Sigma_{g,b}(g \geq 2, b \geq 1)$. For $b = 1$, Johnson gave a full description of this image as $\wedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$. This paper focuses on the case for $b \geq 2$. To compute this, we use the fact that the Torelli group for $\Sigma_{g,b}(g \geq 2, b \geq 1)$ is generated by the Dehn twists about nonseparating curves and bounding pair maps which can be proved by induction and the well-known fact that $\mathscr{T}(\Sigma_{g,0})$ is generated by the Dehn twists about nonseparating curves and bounding pair maps due to Johnson. We give a proof of this theorem in this paper which is due to Justin Malestein. Thus by computing the images of the Dehn twists about nonseparating curves and bounding pair maps, along with the equivariance of the Johnson homomorphism, the whole image of the Johnson homomorphism can be decided. An explicit description in the matrix form is refered to Theorem 4.4 and Theorem 5.1 for $b = 2$ and $b \geq 3$ respectively in this paper. For $b = 2$, the image has a more intrinsic description as the image of $\wedge^3 H_1(\Sigma_{g,2};\mathbb{Z})$ refered to Theorem 4.5. Other than that, we are also interested in the image of the Johnson homomorphism for surfaces of infinite-type, whose fundamental groups are not finitely generated, and we compute a simplest one whose surface has one boundary component, infinite surface and one and only one ends accumulated by genus.

The outline of this paper is as follows. We give an definition of the Torelli groups for compact surfaces in section 2 and show that the Torelli group we defined is generated by all Dehn twists about separating curves and all bounding pair maps; in section 3, we give an introduction of the definition of the Johnson homomorphism and its naturality; in section 4 and 5, we respectively compute the image of the Johonson homomorphism for $\Sigma_{g,2}$ and $\Sigma_{g,b}(b \geq 3)$; at last, in section 6, we compute the image for one special surface of infinite-type.

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2 Torelli Groups for Compact Surfaces

The action of $\mathrm{Mod}(\Sigma_{g,b})$ on $H_1(\Sigma_{g,b};\mathbb{Z})$ induces a representation: $\Psi : \mathrm{Mod}(\Sigma_{g,b}) \to \mathrm{Aut}(H_1(\Sigma_{g,b};\mathbb{Z}))$. The Torelli group for $\Sigma_{g,b}$ is defined as $\mathscr{T}(\Sigma_{g,b}) := \mathrm{Ker}(\Psi)$.
First we discuss some elements in $\mathscr{T}(\Sigma_{g,b})$:

- Dehn twists about separating curves

  Given any simple closed separating curve α in $\Sigma_{g,b}$, there exists a basis $\{\alpha_1,\beta_1,\cdots,\alpha_g,\beta_g,\gamma_1,\gamma_2,\cdots,\gamma_{b-1}\}$ of $\mathrm{H}_1(\Sigma_{g,b};\mathbb{Z})$ where all $\alpha_i,\beta_i,\gamma_i$ are represented by oriented simple closed curves disjoint from α, and since T_α fixes all these curves, it fixes their corresponding homology classes and hence lies in $\mathscr{T}(\Sigma_{g,b})$. Or we can apply the following lemma, using the fact that α have zero algebraic intersection number with all other curves.

- Bounding pair maps

  A bounding pair is a pair of disjoint, homologous, nonisotopic, nonseparating simple closed curves α,β. And we require that the union of α and β separates the surface into two connected components. A bounding pair map refers to $T_\alpha T_\beta^{-1}$ where α and β form a bounding pair.

To show that $T_\alpha T_\beta^{-1}$ lies in $\mathscr{T}(\Sigma_{g,b})$ for arbitrary $\{g,b\}$, we need the following lemma.

Lemma 2.1. *Let α and β be the isotopy classes of two oriented simple closed curves respectively in $\Sigma_{g,b}$. Then for any integer $k \geq 0$, we have $\Psi(T_\beta^k)([\alpha]) = [\alpha] + k\cdot\hat{i}(\alpha,\beta)[\beta]$, where $\hat{i}(\alpha,\beta)$ is the algebraic intersection number.*

Proof of lemma: The lemma is a slight extension of Proposition 6.3 on [2] and the proof is similar.

First for the case where β is separating, by the change of coordinates principle there is a geometric basis $\{\alpha_1,\beta_1,\cdots,\alpha_g,\beta_g,\gamma_1,\gamma_2,\cdots,\gamma_{b-1}\}$ such that each curve is disjoint from β. The lemma follows immediately.

Then assuming that β is nonseparating, by the change of coordinates principle there is a geometric basis $\{\alpha_1,\beta_1,\cdots,\alpha_g,\beta_g,\gamma_1,\gamma_2,\cdots,\gamma_{b-1}\}$ such that $\beta=\beta_1$. Here $i(\alpha_i,\beta_j)=\delta_{ij}$, $i(\alpha_i,\alpha_j)=i(\alpha_i,\gamma_j)=i(\beta_i,\beta_j)=i(\beta_i,\gamma_j)=i(\gamma_i,\gamma_j)=0$, where $i(\alpha,\beta)$ is the geometric intersection number between curve α and curve β. In this way we have

$$\Psi(T_\beta^k)[c] = [T_\beta^k(c)] = \begin{cases} [c], & c\in\{\beta_1,\cdots,\alpha_g,\beta_g,\gamma_1,\gamma_2,\cdots,\gamma_{b-1}\}; \\ [\alpha_1]+k\cdot[\beta], & c=\alpha_1. \end{cases}$$

Now for arbitrary α, the α_1-coefficient of $[\alpha]$ is $\hat{i}(\alpha,\beta)$. By the linearity of $\Psi(T_\beta^k)$, the lemma follows. □

We can see the above lemma that for a bounding pair α,β, the image of T_α and T_β under Ψ are equal, thus $T_\alpha T_\beta^{-1}$ is in $\mathscr{T}(\Sigma_{g,b})$.

Further we have the fact that Dehn twists about separating curves and bounding pair maps generates $\mathscr{T}(\Sigma_{g,b})$, which can be proved by induction from the well-known fact that the Torelli group for $\Sigma_{g,1}$ is generated by Dehn twists about separating curves and bounding pair maps. The proof is due to Justin Malestein.

Theorem 2.2 (Malestein)**.** *For $g \geq 2, b \geq 1, \mathscr{T}(\Sigma_{g,b})$ is generated by Dehn twists about separating curves and bounding pair maps.*

First we need the following lemma which is Lemma A.1 in [9] and Fact 4.7 in [2]. The definition of point-pushing map can be found on Section 4.2.1 in [2]. Notice that in the following we denote by $\Sigma_{g,b,n}$ a connected orientable surface of genus g with b boundary components and n punctures.Then $\Sigma_{g,b} = \Sigma_{g,b,0}$.

Lemma 2.3 (Putman)**.** *For $g \geq 2, b \geq 2$,assume that the base point x is in the interior of $\pi_1(\Sigma_{g,b-1,0})$. Then $[\pi_1(\Sigma_{g,b-1,0},x), \pi_1(\Sigma_{g,b-1,0},x)]$ is generated by $[\gamma_1, \gamma_2]$, where $\gamma_1, \gamma_2 \in \pi_1(\Sigma_{g,b-1,0},x)$ are simple closed curves so that $\gamma_1 \bigcap \gamma_2 = \{x\}$ and so that a regular neighborhood of $\gamma_1 \bigcup \gamma_2$ is homeomorphic to a one-holed torus.*

Fact 2.4 (Farb&Margalit)**.** *Let α be a simple loop which represents an element in $\pi_1(\Sigma_{g,b-1,0})$. Let Push be the point-pushing map: $Push : \pi_1(\Sigma_{g,b-1,0},x) \to Mod(\Sigma_{g,b-1,1})$. Then $Push([\alpha]) = T_\beta T_\gamma^{-1}$, where β and γ are the isotopy classes of the simple closed curves in $\Sigma_{g,b-1,1}$ obtained by pushing α off itself to the left and right respectively.*

Proof of theorem: By capping a boundary component β of $\Sigma_{g,b,0}$ with a once-marked disk, we get an exact sequence(Proposition 3.19 in [2]):

$$1 \to \langle T_\beta \rangle \to \mathrm{Mod}(\Sigma_{g,b,0}) \to \mathrm{Mod}(\Sigma_{g,b-1,1}) \to 1 \tag{1}$$

Since β is separating, T_β is in $\mathscr{T}(\Sigma_{g,b,0})$, so the above exact sequence descends to

$$1 \to \langle T_\beta \rangle \to \mathscr{T}(\Sigma_{g,b,0}) \to \mathscr{T}(\Sigma_{g,b-1,1}) \to 1 \tag{2}$$

We also have the Birman exact sequence

$$1 \to \pi_1(\Sigma_{g,b-1,0},\ x) \xrightarrow{Push} \mathrm{Mod}(\Sigma_{g,b-1,1}) \xrightarrow{Forget} \mathrm{Mod}(\Sigma_{g,b-1,0}) \to 1 \tag{3}$$

where x is a marked point and the forgetful map $Forget : \mathrm{Mod}(\Sigma_{g,b-1,1}) \xrightarrow{Forget} \mathrm{Mod}(\Sigma_{g,b-1,0})$ is realized by forgetting the point x is marked.
The Birman exact sequence descends to

$$1 \to K \xrightarrow{Push} \mathscr{T}(\Sigma_{g,b-1,1}) \xrightarrow{Forget} \mathscr{T}(\Sigma_{g,b-1,0}) \to 1 \tag{4}$$

where $K = \pi_1(\Sigma_{g,b-1,0},x) \bigcap Push^{-1}(\mathscr{T}(\Sigma_{g,b-1,1}))$.

We explain why the map $\mathscr{T}(\Sigma_{g,b-1,1}) \xrightarrow{Forget} \mathscr{T}(\Sigma_{g,b-1,0})$ is surjective. By induction, $\mathscr{T}(\Sigma_{g,b-1,0})$ is generated by Dehn twists about separating curves and bounding pair maps. First let α be a separating curve in $\Sigma_{g,b-1,0}$, so $T_\alpha \in \mathscr{T}(\Sigma_{g,b-1,0})$. If the marked point x is disjoint with the curve α, then naturally α can be embedded in $\Sigma_{g,b-1,1}$ as $\tilde{\alpha}$ which is also separating, so $T_{\tilde{\alpha}} \in \mathscr{T}(\Sigma_{g,b-1,1})$ and $Forget(T_{\tilde{\alpha}}) = T_\alpha$; the other case is that x is a point in α, but we can modify α by isotopy so that it is disjoint with x, then following the same procedure we know that T_α is an image of $\mathscr{T}(\Sigma_{g,b-1,1}) \xrightarrow{Forget} \mathscr{T}(\Sigma_{g,b-1,0})$. Next let $\{\beta, \gamma\}$ be a bounding pair in $\Sigma_{g,b-1,0}$, then $T_\beta T_\gamma^{-1} \in \mathscr{T}(\Sigma_{g,b-1,0})$. No matter the reletive position of x and $\{\beta,\gamma\}$, we can always modify β and γ by isotopy such that new $\{\beta,\gamma\}$ separates $\Sigma_{g,b-1,0}$ into two surfaces, one of which have only β and γ as its boundary components and do not contain the point x. We can still embed β and γ in $\Sigma_{g,b-1,1}$ as $\tilde{\beta}$ and $\tilde{\gamma}$ so that $\{\tilde{\beta},\tilde{\gamma}\}$ forms a bounding pair in $\Sigma_{g,b-1,1}$. Thus $T_{\tilde{\beta}}T_{\tilde{\gamma}}^{-1} \in \mathscr{T}(\Sigma_{g,b-1,1})$ and $Forget(T_{\tilde{\beta}}T_{\tilde{\gamma}}^{-1}) = T_{\beta}T_{\gamma}^{-1}$. In conclusion, each of Dehn twists about separating curves and bounding pair maps in $\Sigma_{g,b-1,0}$ is an image of a Dehn twist about a separating curve or a bounding pair map in $\Sigma_{g,b-1,0}$, so we prove the surjectivety.

Next, we try to find out what K is exactly. Firstly, Proposition 6.13 in [1] tells us for $b=1$, $K=\pi_1(\Sigma_{g,0,0})$; for $b\geq 2$, we claim that K is generated by $\{$separating simple closed curves in $\Sigma_{g,b-1,0}$ based at $x\}$ and we prove our claim as follows.

First, according to Fact 2.4, it's easy to see that $\{$separating simple closed curves in $\Sigma_{g,b-1,0}$ based at $x\} \subset K$ since we know Dehn twists about separating curves lie in the Torelli group.

Take $\{\alpha_1,\beta_1,\cdots,\alpha_g,\beta_g,\gamma_1,\gamma_2,\cdots,\gamma_{b-2}\}$ as a basis of $\Gamma=\pi_1(\Sigma_{g,b-1,0},\ x)$ where $\alpha_i,\ \beta_i$ are non-separating loops with $i(\alpha_i,\beta_j)=\delta_{ij},\ i(\alpha_i,\alpha_j)=i(\beta_i,\beta_j)=0$, and each of $\gamma_1,\cdots,\gamma_{b-2}$ is a loop around one boundary component. For simple closed curves μ_1 and μ_2 such that $\mu_1\bigcap\mu_2=\{x\}$ and the regular neighborhood of $\mu_1\bigcup\mu_2$ is a one-holed torus, so $[\mu_1,\mu_2]$ is a separating curve around the boundary of this one-holed torus. By Fact 2.4 $Push([\mu_1,\mu_2])$ is a pair of Dehn twists about two separating curves. Thus $Push([\mu_1,\mu_2]) \in \mathscr{T}(\Sigma_{g,b-1,1})$ by lemma 2.3. So we know $[\pi_1(\Sigma_{g,b-1,0},\ x),\pi_1(\Sigma_{g,b-1,0},x)] \subset K$. Since the curves obtained by pushing $\gamma_i (1\leq i\leq b-2)$ off itself to the left and right are still separating, we have $\{\gamma_1,\gamma_2,\cdots,\gamma_{b-2}\}\subset K$. Thus if we denote as Γ' the group generated by $[\pi_1(\Sigma_{g,b-1,0},\ x),\pi_1(\Sigma_{g,b-1,0},\ x)]$ and $\{\gamma_1,\gamma_2,\cdots,\gamma_{b-2}\}$, we get $\Gamma'\subset K$.

Next, to prove $K\subset\Gamma'$, instead we can show that $\pi_1(\Sigma_{g,b-1,0},\ x)\backslash\Gamma' \subset \pi_1(\Sigma_{g,b-1,0},\ x)\backslash K$. Let S be the new surface homeomorphic to $\Sigma_{g,0,0}$ obtained by capping all boundary components of $\Sigma_{g,b-1,0}$, then $H_1(S)=\Gamma/\Gamma'$. For any α in $\pi_1(\Sigma_{g,b-1,0},\ x)\backslash\Gamma'$, Suppose $[\alpha]$ is the corresponding element in Γ/Γ' of α, then there exists a primitive element $[\delta]\in\Gamma/\Gamma'$ and an integer m such that $[\alpha]=m\cdot[\delta]$. Due to Prop. 6.2 in [1], $[\delta]$ can be represented by an oriented simple closed curve δ in S based at x. Additionally δ is nonseparating since $[\delta]$ is nontrivial in $H_1(S)$ and S is closed. Take $\tilde{\delta}$ as the preimage of δ under the map $\pi_1(\Sigma_{g,b-1,0},\ x)\to\pi_1(S,x)$ which is induced by capping all boundary components of $\pi_1(\Sigma_{g,b-1,0},\ x)$. $\tilde{\delta}$ is nonseparating as δ is. Then $\alpha=\tilde{\delta}^m\cdot\lambda$ for some $\lambda\in\Gamma'$, so if $\alpha\in K$ we will have $\tilde{\delta}\in K$, which is, however, not the truth. Since $\tilde{\delta}$ is a nonseparating simple closed curve in $\Sigma_{g,b,0}$, there exists some other nonseparating simple closed curve η in $\Sigma_{g,b,1}$ which intersects with it exactly once. Suppose a and b are the curves obtained by pushing $\tilde{\delta}$ off itself to the left and right, then $\Psi((Push(\tilde{\delta})[\eta]=(\Psi(Push(\tilde{\delta}))[\eta])=(\Psi(T_aT_b^{-1})[\eta])=[\eta]+[a]-[b]$. With $b\geq 2$, $[a]\neq[b]$ in $H_1(\Sigma_{g,b,1})$ since $\{a,b\}$ separates $\Sigma_{g,b,1}$ into one surface homeomorphic to $\Sigma_{0,2,1}$ and the other surface homeomorphic to $\Sigma_{g-1,b+2,0}$. It turns out that $Push(\tilde{\delta})\notin\mathscr{T}(\Sigma_{g,b,1})$. Hence δ^m is not in K, so neither is α. So we have shown that everything not in $\Gamma]$ is not in K, so $K\subset\Gamma'$.

Now we have $K=\Gamma'$. While by definition we know that Γ' is generated by separating simple closed curves in $\pi_1(\Sigma_{g,b-1,0})$ based at x, our claim is proven.

Now back to our main theorem, we prove it by induction on the number of boundary components b.

For $b=1$, the result for $\mathscr{T}(\Sigma_{g,1,0})$ comes from $(2),(4)$ where $b=1$ and elements in $Push(\pi_1(\Sigma_{g,0,0},x))$ are all bounding pair maps, together with the fact that $\mathscr{T}(\Sigma_{g,0,0})$ is generated by Dehn twists about separating curves and bounding pair maps.

For $b\geq 2$, assuming the result for $\mathscr{T}(\Sigma_{g,b-1})$ is true, in light of two exact sequences (2) and (4), and knowing that $Push(K)$ consists of products of Dehn twists about separating curves, we can deduce that $\mathscr{T}(\Sigma_{g,b,0})$ is generated by Dehn twists about separating curves and bounding pair maps. So it's done. □

3 Johnson Homomorphism for Compact Surfaces

In this section, we give a general definition of Johnson Homomorphism for $\Sigma_{g,b}$ and explain its naturality.

First we have a lower central series: $\Gamma = \Gamma_1 = \pi_1(\Sigma_{g,b}), \Gamma_i = [\Gamma, \Gamma_{i-1}] for\ i \geq 2$. For convenience, let $H = \Gamma_1/\Gamma_2,\ N = \Gamma_2/\Gamma_3,\ E = \Gamma_1/\Gamma_3$. Then there is an exact sequence:

$$1 \to N \to E \to H \to 1 \tag{5}$$

The following result is a generalization of Lemma2A, Lemma2B and Lemma2C in [4] from $\Sigma_{g,1}$ to $\Sigma_{g,b}$, and the proof is basically the same.

Proposition 3.1 (Johnson)**.** *Let $f \in \mathscr{T}(\Sigma_{g,b})$, f induces a well-defined map of sets:*

$$t_f : E \to N$$
$$x \mapsto f_*(x)x^{-1}$$

Furthermore, t_f induces a well-defined homomorphism $\tau_f : H \to N$. Then the Johnson homomorphism $\tau : \mathscr{T}(\Sigma_{g,b}) \to Hom(H,N)$ defined as $\tau(f) = \tau_f$ is a homomorphism.

Further, we have a natural isomorphism according to Lemma1B in [4]

$$N \cong \wedge^2 H$$
$$[a,b] \leftrightarrow a \wedge b$$

So the Johnson homomorphism can be viewed as a map $\tau : \mathscr{T}(\Sigma_{g,b}) \to Hom(H, \wedge^2 H)$. In the following, we just regard N and $\wedge^2 H$ as equal.

Naturality. Next, we treat the naturality of Johnson homomorphism carefully, by which we mean it's consistent with the actions of $Mod(\Sigma_{g,b})$ on both sides.

To be exact, given $f \in \mathscr{T}(\Sigma_{g,b})$and $\tau_f \in Hom(H,N)$, $g \in Mod(\Sigma_{g,b})$ acts on f and τ_f in the following way:

Let g act on f, we get $g.f = gfg^-1$;

Let g act on τ_f, we get $(g.\tau_f)(x) = g_*(\tau_f(g_*^{-1}(x)))$ for any $x \in H$.

Then we only need to check that $\tau_{gfg^{-1}} = g_*(\tau_f)$, see [6] or [4].

4 The Image of Johnson Homomorphism for $\Sigma_{g,2}$

First we make it clear how to compute the image of Johnson homomorphism for $\Sigma_{g,b}$. Then we follow this way to compute the image for $\Sigma_{g,2}$ and $\Sigma_{g,b}$ for any $b \geq 3$ in this section and next section, repectively.

By Theorem 2.2, we know that $\tau(\mathscr{T}(\Sigma_{g,b}))$ is spanned by $\{\tau(T_\alpha), \tau(T_\beta T_\gamma^{-1})\}$,where α ranges over all an arbitrary separating simple closed curves in $\Sigma_{g,b}$, and $\{\beta, \gamma\}$ ranges over all bounding pairs in $\Sigma_{g,b}$. By the change of coordinates principle, we only need to calculate the images of some particular

Dehn twists about separating curves and bounding pair maps due to the fact that $fT_\alpha f^{-1} = T_{f(\alpha)}$ and $fT_\beta T_\gamma^{-1} f^{-1} = fT_\beta f^{-1} f T_\gamma^{-1} f^{-1} = T_{f(\beta)} T_{f(\gamma)}^{-1}$.
First of all, for $b = 1$, let $H = H_1(\Sigma_{g,1})$. The case $b = 1$ special since we have a canonical isomorphism $H^* \cong H$ induced by the nondegenerate bilinear quadratic form which is the algebraic intersection pairing. However, notice that for $\Sigma_{g,b}$ $(b \geq 2)$ we don't have the natural isomorphism of $H_1(\Sigma_{g,b})$ to its dual since the algebraic intersection pairing is degenerate. Thus especially in the case of $b = 1$ we have: $Hom(H, \wedge^2 H) \cong H^* \otimes \wedge^2 H \cong H \otimes \wedge^2 H$. And $\wedge^3 H$ embeds in $H \otimes \wedge^2 H$ naturally by mapping $a \wedge b \wedge c$ to $a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$. And the image of Johnson homomorphism in this case has been thoroughly discovered by the following proposition 6.16 on [1].

Proposition 4.1 (Johnson)**.** *For $g \geq 2$, $\tau(\mathscr{T}(\Sigma_{g,1})) = \wedge^3 H_1(\Sigma_{g,1})$.*

Next, we try to find the total image for $b = 2$. First, we calculate the images of some particular Dehn twists about separating curves and bounding pair maps.
(a) Dehn twists about separating curves

Separating curves in $\Sigma_{g,2}$ can be classified into two kinds: a separating curve α that separates $\Sigma_{g,2}$ into a surface homeomorphic to $\Sigma_{k,1} (1 \leq k \leq g)$ and a surface homeomorphic to $\Sigma_{g-k,3}$; another kind is a separating curve β that separates $\Sigma_{g,2}$ into a surface homeomorphic to $\Sigma_{k,2} (1 \leq k \leq g)$ and a surface homeomorphic to $\Sigma_{g-k,2}$(See figure 1 below).

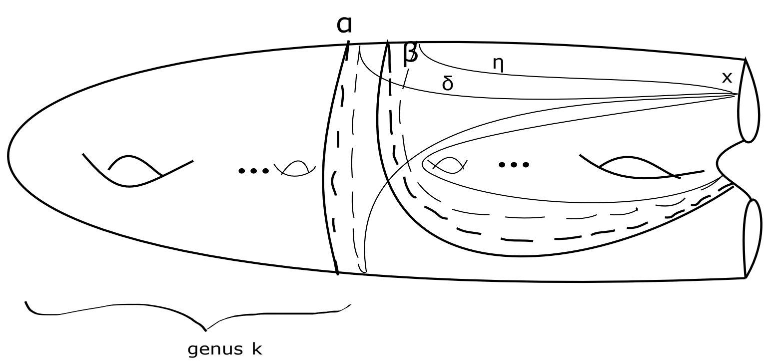

Figure 1: *separating curves*

We can directly compute their image under Johnson homomorphism. First we choose a base point x in one boundary component of $\Sigma_{g,2}$, and take a basis $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1\}$ of $\pi_1(\Sigma_{g,2},\ x)$ (See Figure 2 below).

Then for the first kind we have:

$$\tau(T_\alpha)([\alpha_i]) = T_\alpha(\alpha_i)\alpha_i^{-1} = \delta\alpha_i\delta^{-1}\alpha_i^{-1} = [\delta,\ \alpha_i], 1 \leq i \leq k$$
$$\tau(T_\alpha)([\beta_i]) = T_\alpha(\beta_i)\beta_i^{-1} = \delta\beta_i\delta^{-1}\beta_i^{-1} = [\delta,\ \beta_i], 1 \leq i \leq k$$
$$\tau(T_\alpha)([c]) = T_\alpha(c)c^{-1} = cc^{-1} = 1, \text{ if } c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g, \gamma_1\}$$

where δ is an element in Γ_1 shown in Figure 1. We know that $\delta = \prod_{i=1}^{k} [\alpha_i,\ \beta_i] \in \Gamma_2$, so $[\delta,\ \alpha_i] \in \Gamma_3$, which means $\tau(T_\alpha)([\alpha_i]) = [\delta,\ \alpha_i] = 1 \in N = \Gamma_2/\Gamma_3$. The same argument applies to $\tau(T_\alpha)(\beta_i)$,so in the end we get $\tau(T_\alpha) = 0$, where 0 means a constant map.

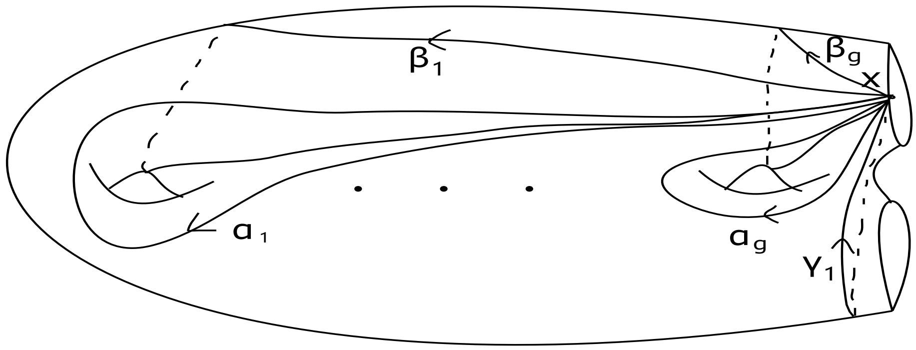

Figure 2: *basis of* $\pi_1(\Sigma_{g,2})$

For the second kind, let $f_k = T_\beta$ where β is a separating curve which separates $\Sigma_{g,2}$ into $\Sigma_{k,2}$ and $\Sigma_{g-k,2}$. We have:

$$\tau(f_k)([\alpha_i]) = T_\beta(\alpha_i)\alpha_i^{-1} = \eta\alpha_i\eta^{-1}\alpha_i^{-1} = [\eta,\ \alpha_i], 1 \le i \le k$$
$$\tau(f_k)([\beta_i]) = T_\beta(\beta_i)\beta_i^{-1} = \eta\beta_i\eta^{-1}\beta_i^{-1} = [\eta,\ \beta_i], 1 \le i \le k$$
$$\tau(f_k)([\gamma_1]) = T_\beta(\gamma_1)\gamma_1^{-1} = \eta\gamma_1\eta^{-1}\gamma_1^{-1} = [\eta,\ \gamma_1]$$
$$\tau(f_k)([c]) = T_\beta(c)c^{-1} = cc^{-1} = 1, \text{ if } c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g\}$$

where η is an element in Γ_1 shown in Figure 1. We have that $[\eta] = [\gamma_1] \in \Gamma_1/\Gamma_2$, so it becomes:

$$\tau(f_k)([\alpha_i]) = [\eta,\ \alpha_i] = [\gamma_1,\ \alpha_i] \leftrightarrow [\gamma_1] \wedge [\alpha_i], 1 \le i \le k$$
$$\tau(f_k)([\beta_i]) = [\eta,\ \beta_i] = [\gamma_1,\ \beta_i] \leftrightarrow [\gamma_1] \wedge [\beta_i], 1 \le i \le k$$
$$\tau(f_k)([c]) = 1 \leftrightarrow 0, \text{ if } c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g, \gamma_1\}$$

(b) Bounding pair maps

Due to the definition of bounding pair maps, any bounding pair α, β must separate $\Sigma_{g,2}$ into a surface homeomorphic to $\Sigma_{k,2}(1 \le k \le g-1)$ and a surface homeomorphic to $\Sigma_{g-k-1,4}$ shown in the figure below.

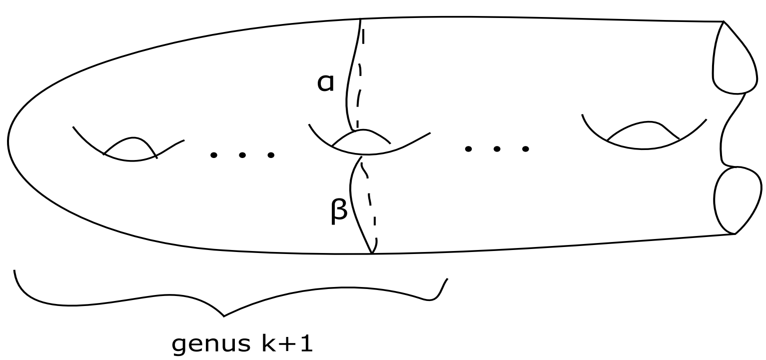

Figure 3: *a bounding pair in* $\Sigma_{g,2}$

Let $g_k = T_\alpha T_\beta^{-1}$, and notice that the process of computing the image of g_k under τ is exactly the same with that in $\Sigma_{g,1}$, so be obtain:

$$\tau(g_k)([\alpha_i]) = [\beta_{k+1}, \alpha_i] \leftrightarrow [\beta_{k+1}] \wedge [\alpha_i], 1 \leq i \leq k$$
$$\tau(g_k)([\beta_i]) = [\beta_{k+1}, \beta_i] \leftrightarrow [\beta_{k+1}] \wedge [\beta_i], 1 \leq i \leq k$$
$$\tau(g_k)([\alpha_{k+1}]) = \prod_{i=1}^{k} [\alpha_i, \beta_i] \leftrightarrow \sum_{i=1}^{k} [\alpha_i] \wedge [\beta_i]$$
$$\tau(g_k)([c]) = 1 \leftrightarrow 0, if\ c \in \{\beta_{k+1}, \alpha_{k+2}, \beta_{k+2}, \cdots, \alpha_g, \beta_g, \gamma_1\}$$

Representation of the image in matrix form. Having taken the bases of $H = H_1(\Sigma_{g,2})$ and $\wedge^2 H$, we could describe any element in $\mathrm{Hom}(H, \wedge^2 H)$ in matrix form like this (For convenience, we denote $c_1 = \alpha_1, c_2 = \alpha_2, \cdots, c_g = \alpha_g, c_{g+1} = \beta_1, \cdots, c_{2g} = \beta_g$):

$$\begin{array}{c} \\ [\gamma_1]\wedge[c_1] \\ \vdots \\ [\gamma_1]\wedge[c_g] \\ [\gamma_1]\wedge[c_{g+1}] \\ \vdots \\ [\gamma_1]\wedge[c_{2g}] \\ [c_i]\wedge[c_j] \end{array} \begin{array}{c} \begin{array}{ccccccc} [\gamma_1] & [c_1] & \dots & [c_g] & [c_{g+1}] & \dots & [c_{2g}] \end{array} \\ \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}_{g(2g+1)\times(2g+1)} \end{array}$$

And the results we have calculated in **(a)** give us only the following nontrivial matrices corresponding to $f_k \in Mod(\Sigma_{g,2})$:

$$\begin{pmatrix} 0_{2g\times 1} & B_k \\ 0_{g(2g-1)\times 1} & 0_{g(2g-1)\times 2g} \end{pmatrix} \tag{6}$$

where

$$B_k = \begin{pmatrix} I_k & & & \\ & 0_{g-k} & & \\ & & I_k & \\ & & & 0_{g-k} \end{pmatrix} \tag{7}$$

Before we do further computation, we need an important lemma below.

Lemma 4.2. *We take a standard basis $\{[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g], [\gamma_1], [\gamma_2], \cdots, [\gamma_{b-1}]\}$ of $H_1(\Sigma_{g,b}; \mathbb{Z})$ shown in Figure 4 below. Then with respect to this basis, the image of $\Psi : Mod(\Sigma_{g,b}) \to Aut(H_1(\Sigma_{g,b}; \mathbb{Z}))$ in matrix form is:*

$$\{P = \begin{pmatrix} \tilde{P}_{2g\times 2g} & 0 \\ v_1 & \\ \vdots & I_{b-1} \\ v_{b-1} & \end{pmatrix} | \forall \tilde{P} \in Sp(2g; \mathbb{Z}), \forall v_i \in M_{1\times 2g}(\mathbb{Z}), 1 \leq i \leq b-1\}$$

In fact, we can describe the result independent of the choice of a basis. The image of $Mod(\Sigma_{g,b})$ can be characterized as those automorphisms of $H_1(\Sigma_{g,b}; \mathbb{Z})$ which:

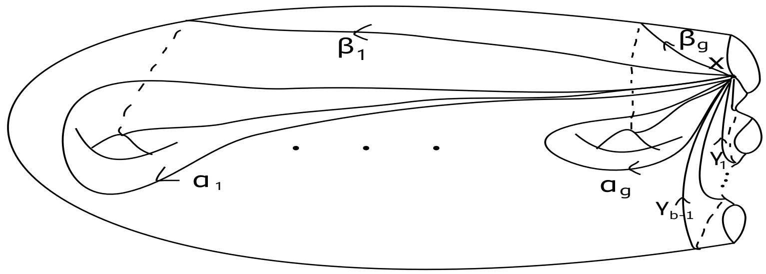

Figure 4: *basis of* $\pi_1(\Sigma_{g,b})$

- *act as identity on the subspace of* $H_1(\Sigma_{g,b};\mathbb{Z})$ *spanned by* $[\gamma_1],[\gamma_2],\cdots,[\gamma_{b-1}]$
- *preserve the algebraic intersection form*

Proof of lemma: To prove this lemma, we need to prove two things. One is to prove every element in the image of Ψ is of the form we want after choosing a basis of $H_1(\Sigma_{g,b};\mathbb{Z})$. The other is to prove every element in $\mathrm{Aut}(H_1(\Sigma_{g,b};\mathbb{Z}))$ in this form is in the image of Ψ.

We first show that every element in the image of Ψ is of this form. On the one hand, since $Mod(\Sigma_{g,b})$ fixes the boundary components pointwise, it doesn't change the homology class of $\gamma_1,\gamma_2,\cdots,\gamma_{b-1}$. On the other hand, the image of $\Psi:\mathrm{Mod}(\Sigma_{g,1})\to\mathrm{Aut}(H_1(\Sigma_{g,1};\mathbb{Z}))$ is $Sp(2g;\mathbb{Z})$, and for any k, $\mathrm{Mod}(\Sigma_{g,k})$ maps surjectively to $\mathrm{Mod}(\Sigma_{g,k-1})$, so by induction we know that $\tilde{P}$ must be symplectic.

What's left to prove is that every element in $\mathrm{Aut}(H_1(\Sigma_{g,b};\mathbb{Z}))$ in this form is in the image of Ψ. First, we oberserve the images of some simple Dehn twists and what their matrices are like.

For $1\le k\le b-1$ and $1\le i\le g$, since $[\beta_i]-[\gamma_k]$ and $-[\alpha_i]-[\gamma_k]$ are primitive in $H_1(\Sigma_{g,b};\mathbb{Z})$, there must be two simple closed curves in $\Sigma_{g,b}$ representing the homology classes $[\beta_i]-[\gamma_k]$ and $-[\alpha_i]-[\gamma_k]$ respectively. Donote them as δ and μ, and we may just take two curves as in Figure 5. Then we have:

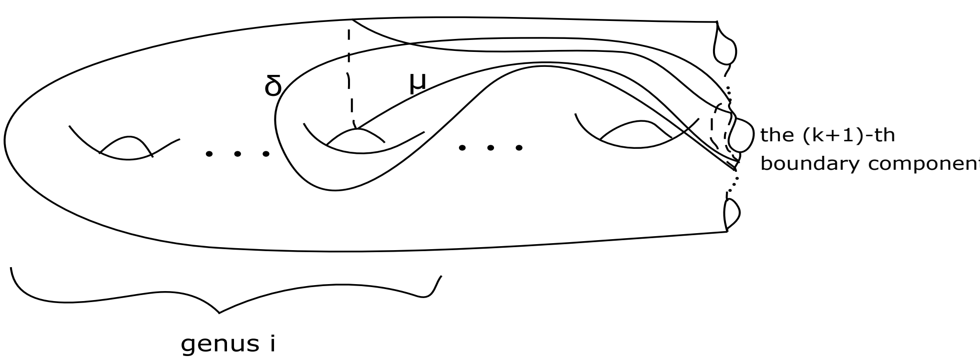

Figure 5: δ *and* μ *in* $\Sigma_{g,b}$

$$\Psi(T_{\beta_i}T_{\delta}^{-1})([\alpha_i]) = [T_{\beta_i}T_{\delta}^{-1}(\alpha_i)] = [\alpha_i] - [\delta] + [\beta_i] = [\alpha_i] + [\gamma_k]$$
$$\Psi(T_{\beta_i}T_{\delta}^{-1})([c]) = [c], \text{ if } c \in \{\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, \gamma_1, \cdots, \gamma_{b-1}\}$$
$$\Psi(T_{\alpha_i}^{-1}T_{\mu}^{-1})([\beta_i]) = [T_{\alpha_i}^{-1}T_{\mu}^{-1}(\beta_i)] = [\beta_i] - [\mu] - [\alpha_i] = [\beta_i] + [\gamma_k]$$
$$\Psi(T_{\beta_i}^{-1}T_{\mu}^{-1})([c]) = [c], \text{ if } c \in \{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_{i-1}, \beta_{i+1}, \cdots, \beta_g, \gamma_1, \cdots, \gamma_{b-1}\}$$

The matrix for $\Psi(T_{\beta_i}T_{\delta}^{-1})$ is determined by $\tilde{P} = I_{2g}$ and $v_1 = \cdots = v_{k-1} = v_{k+1} = \cdots = v_{b-1} = 0, v_k = e_i$ where $e_i = (0, \cdots, 1, \cdots, 0)$ which is the i-th unit vector in $\mathbb{Z}_{2g}$. The matrix for $\Psi(T_{\alpha_i}^{-1}T_{\mu}^{-1})$ is determined by $\tilde{P} = I_{2g}$ and $v_1 = \cdots = v_{k-1} = v_{k+1} = \cdots = v_{b-1} = 0, v_k = e_{g+i}$. Since Ψ is a homomorphism, matrices we get by multiplying what we already got many times should still lie in the image. Thus for any $\{v_1, \cdots, v_{b-1}\}$, there exists an element in the image whose matrix is:

$$\begin{pmatrix} I_{2g} & 0 \\ v_1 & \\ \vdots & I_{b-1} \\ v_{b-1} & \end{pmatrix}$$

Furthermore, since for any $2 \leq k \leq b$, $H_1(\Sigma_{g,k-1}) = H_1(\Sigma_{g,k})/\langle\gamma_{k-1}\rangle$ and $\mathrm{Mod}(\Sigma_{g,k})$ maps surjectively to $\mathrm{Mod}(\Sigma_{g,k-1})$ and for $b = 1$ the image of Ψ is precisely $Sp(2g;\mathbb{Z})$, we know that for any $\tilde{P} \in Sp(2g;\mathbb{Z})$, there exists an element in the image of Ψ whose correponding matrix restricted in the subspace spanned by $\{[\alpha_1], \cdots, [\alpha_g], [\beta_1], \cdots, [\beta_g]\}$ is $\tilde{P}$, which is like:

$$\begin{pmatrix} \tilde{P} & 0 \\ w_1 & \\ \vdots & I_{b-1} \\ w_{b-1} & \end{pmatrix}$$

Then finally the result comes from:

$$\begin{pmatrix} \tilde{P} & 0 \\ v_1 & \\ \vdots & I_k \\ v_{b-1} & \end{pmatrix} = \begin{pmatrix} \tilde{P} & 0 \\ w_1 & \\ \vdots & I_k \\ w_{b-1} & \end{pmatrix} \begin{pmatrix} I_{2g} & 0 \\ v_1 - w_1 & \\ \vdots & I_k \\ v_{b-1} - w_{b-1} & \end{pmatrix}$$

So far we have proven that every element in this form is in the image of Ψ. Combining two sides, we have proven the lemma.

□

Then we can continue our computation by letting $\varphi \in Mod(\Sigma_{g,2})$ act on $f_k \in Mod(\Sigma_{g,2})$ whose corresponding matrix is (6). Supposing the matrix of φ is P, we will get the inverse of P:

$$P^{-1} = \begin{pmatrix} \tilde{P}^{-1} & 0 \\ w_1 & 1 \end{pmatrix}$$

where $w_1 = -v_1 \cdot \tilde{P}^{-1}$. If we denote $A_{i,j}$ as the item in the i-th row and the j-th column of any matrix A, and denote v^i as the i-th coordinate of a vector v, we can compute out:

$$\begin{aligned}
\varphi_*(f_k)([c_i]) &= \varphi_*(f_k(\varphi_*^{-1}([c_i]))) \\
&= \varphi_*(f_k(\sum_{j=1}^{2g}(\tilde{P}^{-1})_{j,i}[c_j] + w_1^i[\gamma_1])) \\
&= \sum_{j=1}^{2g}(\tilde{P}^{-1})_{j,i}\varphi_*(f_k([c_j])) \\
&= \sum_{j=1}^{2g}(\tilde{P}^{-1})_{j,i}\varphi_*(\sum_{l=1}^{2g}(B_k)_{l,j}[\gamma_1]\wedge[c_l]) \\
&= \sum_{j=1}^{2g}\sum_{l=1}^{2g}(\tilde{P}^{-1})_{j,i}(B_k)_{l,j}\varphi_*([\gamma_1])\wedge\varphi_i([c_l]) \\
&= \sum_{j=1}^{2g}\sum_{l=1}^{2g}\sum_{t=1}^{2g}(\tilde{P}^{-1})_{j,i}(B_k)_{l,j}\tilde{P}_{t,l}[\gamma_1]\wedge[c_t]
\end{aligned}$$

$$\varphi_*(f_k)([\gamma_1]) = \varphi_*(f_k(\varphi_*^{-1}([\gamma_1]))) = \varphi_*(f_k([\gamma_1])) = 0$$

Thus the matrix form of $\varphi_*(f_k)$ is like:

$$\begin{pmatrix} 0 & \tilde{P}B_k\tilde{P}^{-1} \\ 0 & 0 \end{pmatrix}$$

Proposition 4.3. *If B_k is a matrix in the form of* (7)*, we have:*

$$\begin{aligned}
&span_{\mathbb{Z}}\{\tilde{P}B_k\tilde{P}^{-1} | 1 \leq k \leq g, \forall \tilde{P} \in Sp(2g;\mathbb{Z})\} \\
=&\{\begin{pmatrix} A & B \\ C & A^t \end{pmatrix} | \forall A,B,C \in M_{g\times g}(\mathbb{Z}), B^t = -B, C^t = -C\}
\end{aligned}$$

Proof of proposition: Let $E_{i,j}$ be the fundamental matrix whose (i,j) entry is 1 and other entries are 0. First we can tranform our target into

$$span_{\mathbb{Z}}\{\tilde{P}\begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix}\tilde{P}^{-1} | 1 \leq k \leq g, \forall \tilde{P} \in Sp(2g;\mathbb{Z})\}$$

For any $\tilde{P} \in Sp(2g;\mathbb{Z})$, it can be written as:

$$\tilde{P} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

And symplectic matrix $\tilde{P}$ satisfies $\tilde{P}^t J \tilde{P} = J$, where

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

so that:

$$\tilde{P}^{-1} = J^{-1}\tilde{P}J = \begin{pmatrix} W^t & -Y^t \\ -Z^t & X^t \end{pmatrix}$$

On the one hand, we can show the left-side set is contained in the right-side set:

$$\begin{aligned} \tilde{P}\begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix}\tilde{P}^{-1} &= \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix}\begin{pmatrix} W^t & -Y^t \\ -Z^t & X^t \end{pmatrix} \\ &= \begin{pmatrix} XE_{k,k}W^t - YT_{k,k}Z^t & -XE_{k,k}Y^t + YE_{k,k}X^t \\ ZE_{k,k}W^t - WE_{k,k}Z^t & WE_{k,k}X^t - ZE_{k,k}Y^t \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} (XE_{k,k}W^t - YT_{k,k}Z^t)^t &= WE_{k,k}X^t - ZE_{k,k}Y^t \\ (-XE_{k,k}Y^t + YE_{k,k}X^t)^t &= -(-XE_{k,k}Y^t + YE_{k,k}X^t) \\ (ZE_{k,k}W^t - WE_{k,k}Z^t)^t &= -(ZE_{k,k}W^t - WE_{k,k}Z^t) \end{aligned}$$

On the other hand, to show the right-side set is contained in the left-side set, firstly we look at some special elements in the right-side set. We know that the following matrices are in $Sp(2g;\mathbb{Z})$:

$$\begin{pmatrix} I+E_{i,j} & 0 \\ 0 & I-E_{j,i} \end{pmatrix}(i \neq j); \qquad \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}(Y^t = Y); \qquad \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix}(Z^t = Z)$$

Then we can compute for the first kind:

$$\begin{aligned} &\begin{pmatrix} I+E_{i,j} & 0 \\ 0 & I-E_{j,i} \end{pmatrix}\begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix}\begin{pmatrix} I+E_{i,j} & 0 \\ 0 & I-E_{j,i} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I+E_{i,j} & 0 \\ 0 & I-E_{j,i} \end{pmatrix}\begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix}\begin{pmatrix} I-E_{i,j} & 0 \\ 0 & I+E_{j,i} \end{pmatrix} \\ &= \begin{pmatrix} (I+E_{i,j})E_{k,k}(I-E_{i,j}) & 0 \\ 0 & (I-E_{j,i})E_{k,k}(I+E_{j,i}) \end{pmatrix} \\ &= \begin{pmatrix} E_{k,k} - \delta_k^i E_{k,j} + \delta_k^j E_{i,j} & 0 \\ 0 & E_{k,k} - \delta_k^i E_{j,i} + \delta_k^j E_{k,i} \end{pmatrix} \end{aligned}$$

Especially, when $k = j$, the above matrix becomes

$$\begin{pmatrix} E_{i,j} & 0 \\ 0 & E_{j,i} \end{pmatrix}$$

which will actually span any matrix like:

$$\begin{pmatrix} A & 0 \\ 0 & A^t \end{pmatrix}$$

The result of the second kind looks like:

$$\begin{aligned}
&\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix} \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} E_{k,k} & \sum_{i=1}^{k-1} Y_{i,k} E_{i,k} - \sum_{j=1}^{k-1} Y_{k,j} E_{k,j} \\ 0 & E_{k,k} \end{pmatrix} \\
&= \begin{pmatrix} E_{k,k} & \sum_{i=1}^{k-1} Y_{i,k} E_{i,k} - \sum_{j=1}^{k-1} Y_{j,k} E_{k,j} \\ 0 & E_{k,k} \end{pmatrix}
\end{aligned}$$

which minus B_k will span any matrix in the form of:

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \quad (B^t = -B)$$

Similarly, we have:

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \begin{pmatrix} E_{k,k} & 0 \\ 0 & E_{k,k} \end{pmatrix} \begin{pmatrix} I & 0 \\ -Z & I \end{pmatrix} \\
&= \begin{pmatrix} E_{k,k} & 0 \\ \sum_{i=1}^{k-1} Z_{i,k} E_{i,k} - \sum_{j=1}^{k-1} Z_{k,j} E_{k,j} & E_{k,k} \end{pmatrix} \\
&= \begin{pmatrix} E_{k,k} & 0 \\ \sum_{i=1}^{k-1} Z_{i,k} E_{i,k} - \sum_{j=1}^{k-1} Z_{j,k} E_{k,j} & E_{k,k} \end{pmatrix}
\end{aligned}$$

which minus B_k will span any matrix in the form of:

$$\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \quad (C^t = -C)$$

So far we've proven that any matrix on the right side can be spanned by elements on the left side together with the other direction, thus our proof is done. $\square$

Then it finally goes to our image of Johnson homomorphism for $\Sigma_{g.2}$.

Theorem 4.4. *If we take the standard basis $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1\}$ of $\pi_1(\Sigma_{g,2}, x)$ (see Figure 2), the image of $\tau : \mathscr{T}(\Sigma_{g,2}) \to Hom(H, \wedge^2 H)$ in the matrix form is:*

$$\{ \begin{pmatrix} 0 & A & B \\ & C & A^t \\ 0 & D & \end{pmatrix} | \forall A, B, C \in M_{g\times g}(\mathbb{Z}), B^t = -B, C^t = -C, \forall D \in \wedge^3 H' \}$$

where $H' = H / \langle \gamma_1 \rangle = H_1(\Sigma_{g,1})$, and $\wedge^3 H'$ has a natural way of embedding in $Hom(H', \wedge^2 H')$.

proof of theorem: Fisrt of all, we know that for $\Sigma_{g,2}$:

$$Im(\tau) = Span_{\mathbb{Z}}\{\varphi_*(f_k), \varphi_*(g_l) | 1 \le k \le g, 1 \le l \le g-1, \forall \varphi \in Mod(\Sigma_{g,2})\}$$

where f_k and g_l are respectively a Dehn twist about a separating curve in **(a)** and a bounding pair map in **(b)**.

Computations before and Proposition 4.3 show us:

$$\begin{aligned} & Span_{\mathbb{Z}}\{\varphi_*(f_k) | 1 \le k \le g, \forall \varphi \in Mod(\Sigma_{g,2})\} \\ = \{ & \begin{pmatrix} 0 & \begin{matrix} A & B \\ C & A^t \end{matrix} \\ 0 & 0 \end{pmatrix} | \forall A, B, C \in M_{g\times g}(\mathbb{Z}), B^t = -B, C^t = -C\} \end{aligned} \tag{8}$$

So in the next step we need to confirm that $\varphi_*(g_k)(1 \le k \le g-1, \forall \varphi \in Mod(\Sigma_{g,2}))$ takes the matrix form in the proposition and we have to show that any matrix in this form is truely an image.

Firstly, we just compute directly $\varphi_*(g_k)(1 \le k \le g-1, \forall \varphi \in Mod(\Sigma_{g,2}))$ like what we did to $\varphi_*(f_k)$, also supposing the matrix of φ is P with the form in Lemma 4.2 for $b = 2$:

$$\varphi_*(g_k)([\gamma_1]) = \varphi_*(g_k(\varphi_*^{-1}([\gamma_1])) = \varphi_*(g_k([\gamma_1])) = 0$$

$$\begin{aligned} \varphi_*(g_k)([c_i]) =& \varphi_*(g_k(\varphi_*^{-1}([c_i]))) \\ =& \varphi_*(g_k(\sum_{j=1}^{2g} (\tilde{P}^{-1})_{j,i}[c_j] + w_1^i[\gamma_1])) \\ =& \varphi_*(g_k(\sum_{j=1}^{k} \tilde{P}_{j,i}^{-1}[\alpha_j])) + \varphi_*(g_k(\tilde{P}_{k+1,i}^{-1}[\alpha_{k+1}])) + \varphi_*(g_k(\sum_{j=g+1}^{g+k} \tilde{P}_{j,i}^{-1}[\beta_{j-g}])) \\ =& \sum_{j=1}^{k} \tilde{P}_{j,i}^{-1}\varphi_*([\beta_{k+1}] \wedge [\alpha_j]) + \tilde{P}_{k+1,i}^{-1}\varphi_*(\sum_{l=1}^{k}[\alpha_l] \wedge [\beta_l]) + \sum_{j=g+1}^{g+k} \tilde{P}_{j,i}^{-1}\varphi_*([\beta_{k+1}] \wedge [\beta_{j-g}]) \\ =& \sum_{j=1}^{k} \tilde{P}_{j,i}^{-1}(v_1^{g+k+1}[\gamma_1] + \sum_{m=1}^{2g} \tilde{P}_{m,g+k+1}[c_m]) \wedge (v_1^j[\gamma_1] + \sum_{n=1}^{2g} \tilde{P}_{n,j}[c_n]) \\ & + \tilde{P}_{k+1,i}^{-1}\sum_{l=1}^{k}(v_1^l[\gamma_1] + \sum_{m=1}^{2g} \tilde{P}_{m,l}[c_m]) \wedge (v_1^{g+l}[\gamma_1] + \sum_{n=1}^{2g} \tilde{P}_{n,g+l}[c_n]) \\ & + \sum_{j=g+1}^{g+k} \tilde{P}_{j,i}^{-1}(v_1^{g+k+1}[\gamma_1] + \sum_{m=1}^{2g} \tilde{P}_{m,g+k+1}[c_m]) \wedge (v_1^j[\gamma_1] + \sum_{n=1}^{2g} \tilde{P}_{n,j}[c_n]) \end{aligned}$$

The part of $[c_m] \wedge [c_n]$ is just the image of $\tilde{\varphi}_*(\tilde{f}_k)$ where $\tilde{\varphi} \in Mod(\Sigma_{g,1})$ corresponds to the matrix $\tilde{P} \in M_{2g\times 2g}(\mathbb{Z})$ and $\tilde{f}_k$ is the image of f_k under $\mathscr{T}(\Sigma_{g,2}) \to \mathscr{T}(\Sigma_{g,1})$. And Proposition 4.1 tells us that this part lies in $\wedge^3 H'$. After we eliminate this part, all the items left have the form of a multiple of $[\gamma_1] \wedge [c_n]$. Denote $Q_{i,n}$ as the coefficient of $[\gamma_1] \wedge [c_n]$ in $\varphi_*(f_k(\varphi_*^{-1}([c_i])))$, then we have:

$$
\begin{aligned}
Q_{i,n} = &\sum_{j=1}^{k} v_1^{g+k+1} \tilde{P}^{-1}_{j,i} \tilde{P}_{n,j} + \sum_{j=g+1}^{g+k} v_1^{g+k+1} \tilde{P}^{-1}_{j,i} \tilde{P}_{n,j} \\
&- \sum_{j=1}^{k} v_1^{j} \tilde{P}^{-1}_{j,i} \tilde{P}_{n,g+k+1} + \sum_{j=1}^{k} v_1^{j} \tilde{P}^{-1}_{k+1,i} \tilde{P}_{n,g+j} \\
&- \sum_{j=1}^{k} v_1^{g+j} \tilde{P}^{-1}_{k+1,i} \tilde{P}_{n,j} - \sum_{j=g+1}^{g+k} v_1^{j} \tilde{P}^{-1}_{j,i} \tilde{P}_{n,g+k+1}
\end{aligned}
$$

Remember that:

$$
\tilde{P} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},\ \tilde{P}^{-1} = J^{-1}\tilde{P}J = \begin{pmatrix} W^t & -Y^t \\ -Z^t & X^t \end{pmatrix}
$$

So we can write $Q_{i,n}$ more specifically:
(i)For $1 \le i \le g, 1 \le n \le g$:

$$
\begin{aligned}
Q_{i,n} = &\sum_{j=1}^{k} v_1^{g+k+1} (W_{i,j}X_{n,j} - Z_{i,j}Y_{n,j}) + \sum_{j=1}^{k} v_1^{j} (-W_{i,j}Y_{n,k+1} + W_{i,k+1}Y_{n,j}) \\
&+ \sum_{j=1}^{k} v_1^{g+j} (-W_{i,k+1}X_{n,j} + Z_{i,j}Y_{n,k+1})
\end{aligned}
$$

(ii)For $1 \le i \le g, g+1 \le n \le 2g$:

$$
\begin{aligned}
Q_{i,n} = &\sum_{j=1}^{k} v_1^{g+k+1} (W_{i,j}Z_{n-g,j} - Z_{i,j}W_{n-g,j}) + \sum_{j=1}^{k} v_1^{j} (-W_{i,j}W_{n-g,k+1} + W_{i,k+1}W_{n-g,j}) \\
&+ \sum_{j=1}^{k} v_1^{g+j} (-W_{i,k+1}Z_{n-g,j} + Z_{i,j}W_{n-g,k+1})
\end{aligned}
$$

(iii)For $g+1 \le i \le 2g, 1 \le n \le g$:

$$
\begin{aligned}
Q_{i,n} = &\sum_{j=1}^{k} v_1^{g+k+1} (-Y_{i-g,j}X_{n,j} + X_{i-g,j}Y_{n,j}) + \sum_{j=1}^{k} v_1^{j} (Y_{i-g,j}Y_{n,k+1} - Y_{i-g,k+1}Y_{n,j}) \\
&+ \sum_{j=1}^{k} v_1^{g+j} (Y_{i-g,k+1}X_{n,j} - X_{i-g,j}Y_{n,k+1})
\end{aligned}
$$

(iv)For $g+1 \le i \le 2g, g+1 \le n \le 2g$:

$$
\begin{aligned}
Q_{i,n} = &\sum_{j=1}^{k} v_1^{g+k+1} (-Y_{i-g,j}Z_{n-g,j} + X_{i-g,j}W_{n-g,j}) + \sum_{j=1}^{k} v_1^{j} (Y_{i-g,j}W_{n-g,k+1} - Y_{i-g,k+1}W_{n-g,j}) \\
&+ \sum_{j=1}^{k} v_1^{g+j} (Y_{i-g,k+1}Z_{n-g,j} - X_{i-g,j}W_{n-g,k+1})
\end{aligned}
$$

And we can easily check that:

①For $1 \leq i \leq g, 1 \leq n \leq g$: $Q_{i,n} = Q_{n+g,i+g}$
②For $1 \leq i \leq g, g+1 \leq n \leq 2g$: $Q_{i,n} = -Q_{n-g,i+g}$
③For $g+1 \leq i \leq 2g, 1 \leq n \leq g$: $Q_{i,n} = -Q_{n+g,i-g}$

All these conditions exactly testify that $\varphi_*(g_k)$ takes the wanted matrix form.

Lastly, we only need to prove that any matrix in this pariticular form could be achieved in the image. Since $\mathscr{T}(\Sigma_{g,2}) \to \mathscr{T}(\Sigma_{g,1})$ is surjective, the following map is also surjective:

$$\begin{pmatrix} 0 & \begin{matrix} A & B \\ C & A^t \end{matrix} \\ 0 & \text{D} \end{pmatrix} \in Im(\tau(\mathscr{T}(\Sigma_{g,2}))) \to D \in Im(\tau(\mathscr{T}(\Sigma_{g,1})))$$

Thus for any $D \in \wedge^3 H'$, there exists an element in $Im(\tau(\mathscr{T}(\Sigma_{g,2})))$ whose matrix descends to D. With (8) and this fact, our proposition is proved. □

To describe the image in a more intrinsic way, we go back to $\wedge^3 H$. Despite the fact that for $b \geq 2$, $\wedge^3 H$ can't be naturally embeded in $\text{Hom}(H, \wedge^2 H)$ as the case for $b = 1$, there is still a natural map from $\wedge^3 H$ to $\text{Hom}(H, \wedge^2 H)$:

1. There is a natural inclusion of $\wedge^3 H$ into $H \otimes \wedge^2 H$ given by $a \wedge b \wedge c \mapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$.
2. There is a map from H to H^* using the algebraic intersection pairing $\hat{i}(-,-)$: $a \in H \mapsto f_a \in H^* : \forall b \in H, f_a(b) = \hat{i}(b,a)$. So there is a map from $H \otimes \wedge^2 H$ to $H^* \otimes \wedge^2 H$.
3. $H^* \otimes \wedge^2 H$ is canonically isomorphic to $\text{Hom}(H, \wedge^2 H)$.

Thus we have a map $\phi : \wedge^3 H \to \text{Hom}(H, \wedge^2 H)$. For $b = 2$, we actually have the following theorem describing the image of Johnson homomorphism independent of matrices.

Theorem 4.5. *The image of $\tau : \mathscr{T}(\Sigma_{g,2}) \to Hom(H, \wedge^2 H)$ is the image of $\phi : \wedge^3 H \to Hom(H, \wedge^2 H)$.*

Proof of theorem. We have mentioned before that $\tau : \mathscr{T}(\Sigma_{g,2}) \to Hom(H, \wedge^2 H)$ is equivariant under the action of $\text{Mod}(\Sigma_{g,2})$. It's easy to check $\phi : \wedge^3 H \to \text{Hom}(H, \wedge^2 H)$ is also equivariant under the action of $\text{Mod}(\Sigma_{g,2})$. Observing ϕ, we only need to check $H \to H^*$ is equivariant under the action of $\text{Mod}(\Sigma_{g,2})$. Let $g \in \text{Mod}(\Sigma_{g,2})$ act on $a \in H$, we get $g.a = g_*(a)$. Let $g \in \text{Mod}(\Sigma_{g,2})$ act on $f_a \in H^*$, we get $(g.f_a)(b) = f_a(g_*^{-1}(b)) = \hat{i}(g_*^{-1}(b), a)$. We may as well take $g = T_c$ where c is a simple closed curve in $\Sigma_{g,2}$ and assume that a, b can be represented as the homology classes of simple closed curves (use the same notation) in $\Sigma_{g,2}$.Then we have $\hat{i}((T_c)_*^{-1}(b), a) = \hat{i}(b - \hat{i}(b,c)c, a) = \hat{i}(b,a) - \hat{i}(c,a)\hat{i}(b,c) = \hat{i}(b, a + \hat{i}(a,c)c) = \hat{i}(b, (T_c)_*(a))$. Thus $(g.f_a)(b) = \hat{i}(b, g_*(a)) = f_{g.a}(b)$, which verifies the equivariance.

We still take a canonical basis $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1\}$ of $\pi_1(\Sigma_{g,2},\ x)$ as in Figure 2.

We need to prove two directions: $Im(\tau) \subset Im(\phi)$ and $Im(\phi) \subset Im(\tau)$.

As for $Im(\tau) \subset Im(\phi)$, it's sufficient to prove that $\tau(f_k)$ and $\tau(g_k)$ computed before in **(a)** and **(b)** lie in the image of ϕ, since both τ and ϕ are equivariant under the action of $\text{Mod}(\Sigma_{g,2})$ and

$$Im(\tau) = Span_{\mathbb{Z}}\{\varphi_*(f_k), \varphi_*(g_l) | 1 \leq k \leq g, 1 \leq l \leq g-1, \forall \varphi \in Mod(\Sigma_{g,2})\}$$

First, for $\tau(f_k)$ where f_k is a Dehn twist about a separating curve shown in **(a)**, we recall that

$$\tau(f_k)([\alpha_i]) = [\gamma_1] \wedge [\alpha_i],\ \tau(f_k)([\beta_i]) = [\gamma_1] \wedge [\beta_i], 1 \le i \le k$$
$$\tau(f_k)([c]) = 0,\ \text{if } c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g, \gamma_1\}$$

We can directly compute $\phi([\gamma_1] \wedge [\beta_1] \wedge [\alpha_1])$ as follows:

$$\begin{aligned}\phi([\gamma_1] \wedge [\alpha_1] \wedge [\beta_1])([c]) &= (f_{[\gamma_1]} \otimes ([\alpha_1] \wedge [\beta_1]) + f_{[\alpha_1]} \otimes ([\beta_1] \wedge [\gamma_1]) + f_{[\beta_1]} \otimes ([\gamma_1] \wedge [\alpha_1]))([c]) \\ &= \hat{i}([c],[\gamma_1])[\alpha_1] \wedge [\beta_1] + \hat{i}([c],[\alpha_1])[\beta_1] \wedge [\gamma_1] + \hat{i}([c],[\beta_1])[\gamma_1] \wedge [\alpha_1]\end{aligned}$$

so we have:

$$\phi([\gamma_1] \wedge [\alpha_1] \wedge [\beta_1])([\alpha_1]) = [\gamma_1] \wedge [\alpha_1],\ \phi([\gamma_1] \wedge [\alpha_1] \wedge [\beta_1])([\beta_1]) = [\gamma_1] \wedge [\beta_1]$$
$$\phi([\gamma_1] \wedge [\alpha_1] \wedge [\beta_1])([c]) = 0,\ \text{if } c \in \{\alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \gamma_1\}$$

Thus $\tau(f_1) = \phi([\gamma_1] \wedge [\alpha_1] \wedge [\beta_1])$. By similar computation, we can get $\tau(f_k) = \phi(\sum_{i=1}^{k} [\gamma_1] \wedge [\alpha_i] \wedge [\beta_i]) \in Im(\phi),\ 1 \le k \le g$.

Then we recall results about $\tau(g_k)$, where g_k is a bounding pair map shown in **(b)**:

$$\tau(g_k)([\alpha_i]) = [\beta_{k+1}] \wedge [\alpha_i],\ \tau(g_k)([\beta_i]) = [\beta_{k+1}] \wedge [\beta_i], 1 \le i \le k$$
$$\tau(g_k)([\alpha_{k+1}]) = \sum_{i=1}^{k} [\alpha_i] \wedge [\beta_i]$$
$$\tau(g_k)([c]) = 0, if\ c \in \{\beta_{k+1}, \alpha_{k+2}, \beta_{k+2}, \cdots, \alpha_g, \beta_g, \gamma_1\}$$

Through basic computation, we have $\tau(g_k) = \phi((\sum_{i=1}^{k} [\alpha_i] \wedge [\beta_i]) \wedge [\beta_{k+1}]) \in Im(\phi), 1 \le k \le g-1$. So far, we have proven the first part.

As for $Im(\phi) \subset Im(\tau)$, we need to show every $\phi(x \wedge y \wedge z)$ is in the image of τ, where $x, y, z \in \{[\alpha_1], [\beta_1], \cdots, [\alpha_g], [\beta_g], [\gamma_1]\}$.

First, from above computation, we know that $\phi([\alpha_1] \wedge [\beta_1] \wedge [\beta_2]) = \tau(g_1) \in Im(\tau)$. Since $Im(\tau)$ is closed under the action of $\mathrm{Mod}(\Sigma_{g,2})$, after we let any $g \in \mathrm{Mod}(\Sigma_{g,2})$ act on $[\alpha_1] \wedge [\beta_1] \wedge [\beta_2]$, its image under ϕ should still lie in the image of τ. Then we can imitate the proof of $Im(\tau) = \wedge^3 H$ for $b = 1$ in Johnson's paper [4]. For $g = 2$, we apply the factor rotation $[\alpha_2] \to [\beta_2] \to -[\alpha_2]$ (basic elements not mentioned are assumed fixed) to $[\alpha_1] \wedge [\beta_1] \wedge [\beta_2]$ and get $\phi(-[\alpha_1] \wedge [\beta_1] \wedge [\alpha_2]) \in Im(\tau)$. The factor swap $[\alpha_1] \leftrightarrow [\alpha_2], [\beta_1] \leftrightarrow [\beta_2]$ applied to these two elements gives us that $\phi([\alpha_2] \wedge [\beta_2] \wedge [beta_1]), \phi(-[\alpha_2] \wedge [\beta_2] \wedge [\alpha_1]) \in Im(\tau)$. If $g \ge 3$, we apply the map $[\alpha_1] \to [\alpha_1] + [\beta_1] - [\beta_3], [\alpha_3] \to [\alpha_3] - [\beta_1] + [\beta_3]$ to $[\alpha_1] \wedge [\beta_1] \wedge [\beta_2]$, and get $\phi([\beta_1] \wedge [\beta_2] \wedge [\beta_3]) \in Im(\tau)$. If we apply to the two elements $[\alpha_1] \wedge [\beta_1] \wedge [\beta_2], [\beta_1] \wedge [\beta_2] \wedge [\beta_3]$ maps of the type 1) $[\alpha] \leftrightarrow [\alpha_i], [\beta_1] \leftrightarrow [\beta_i]$, 2) $[\alpha_i] \to [\beta_i] \to -[\alpha_i]$, we will get all $\phi(x \wedge y \wedge z) \in Im(\tau)$, where $x, y, z \in \{[\alpha_1], [\beta_1], \cdots, [\alpha_g], [\beta_g]\}$.

Then what's left to prove is that all $\phi([\gamma_1] \wedge x \wedge y) \in Im(\tau)$, where $x, y \in \{[\alpha_1], [\beta_1], \cdots, [\alpha_g], [\beta_g]\}$. From above, we have known that $\phi([\gamma_1] \wedge [\alpha_1] \wedge [\beta_1]) = \tau(f_1) \in Im(\tau), \phi([\gamma_1] \wedge [\alpha_i] \wedge [\beta_i]) = \tau(f_i) - \tau(f_{i-1}) \in Im(\tau), 2 \le i \le g$. We apply the factor mix $[\alpha_i] \to [\alpha_i] - [\beta_j], [\alpha_j] \to [\alpha_j] - [\beta_i], (i \ne j)$ to $[\gamma_1] \wedge [\alpha_i] \wedge [\beta_i]$, and get $\phi([\gamma_1] \wedge [\beta_i] \wedge [\beta_j]) \in Im(\tau)$. Then we apply the factor rotation $[\alpha_i] \to [\beta_i] \to -[\alpha_i]$ to $[\gamma_1] \wedge [\beta_i] \wedge [\beta_j]$, and get $\phi(-[\gamma_1] \wedge [\alpha_i] \wedge [\beta_j]) \in Im(\tau)$. Finally we apply $[\alpha_j] \to [\beta_j] \to -[\alpha_j]$ to $-[\gamma_1] \wedge [\alpha_i] \wedge [\beta_j]$ and get $\phi([\gamma_1] \wedge [\alpha_i] \wedge [\alpha_j]) \in Im(\tau)$.

Conclusively we have proven that $Im(\tau) = Im(\phi)$.

□

5 The Image of Johnson Homomorphism for $\Sigma_{g,b}(b \geq 3)$

We continue to compute the image for $\Sigma_{g,b}(b \geq 3)$ following basically the same framework in the previous section. We take a marked point x in one of the boundary components of $\Sigma_{g,b}$, and a standard basis $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, \gamma_1, \gamma_2, \cdots, \gamma_{b-1}\}$ of $\pi_1(\Sigma_{g,b}, x)$ shown in Figure 4. We call the boundary component whose homology class is $[\gamma_i]$ the $(i+1)$-th boundary component. We denote $H = H_1(\Sigma_{g,b};\mathbb{Z})$, $H' = H_1(\Sigma_{g,1};\mathbb{Z})$, and for convenience, we denote $c_1 = \alpha_1, c_2 = \alpha_2, \cdots, c_g = \alpha_g, c_{g+1} = \beta_1, \cdots, c_{2g} = \beta_g$. Then we can describe an element in $Hom(H, \wedge^2 H)$ in the following matrix form:

$$
\begin{array}{c}
\\
[\gamma_i]\wedge[\gamma_j] \\
[\gamma_1]\wedge[c_i] \\
[\gamma_2]\wedge[c_i] \\
\vdots \\
[\gamma_{b-1}]\wedge[c_i] \\
[c_i]\wedge[c_j]
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
[\gamma_1] & \dots & [\gamma_{b-1}] & [c_1] & \dots & [c_g] & [c_{g+1}] & \dots & [c_{2g}]
\end{array}\\
\left(\begin{array}{ccccccccc}
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & *
\end{array}\right)_{(C^2_{b-1}+2g(b-1)+C^2_{2g})\times(2g+b-1)}
\end{array}
$$

First we compute all images of standard Dehn twists about separating curves and bounding pair maps:

(a) Dehn twists about separating curves

In $\Sigma_{g,b}(b \geq 3)$, there are more types of separating curves. Basically, a separating curve μ_k^t separates $\Sigma_{g,b}$ into a surface homeomorphic to $\Sigma_{k,t+1}(0 \leq k \leq g, 0 \leq t \leq b-1)$ and the other surface homeomorphic to $\Sigma_{g-k,b-t+1}$, which contained the marked point x.

First, for μ_k^0, it's easy to get $\tau(T_{\mu_k^0}) = 0$ like before.

Second, for μ_k^1, supposing this curve separates the $(i+1)$-th boundary component from all the other boundary components in the figure below, we will have:

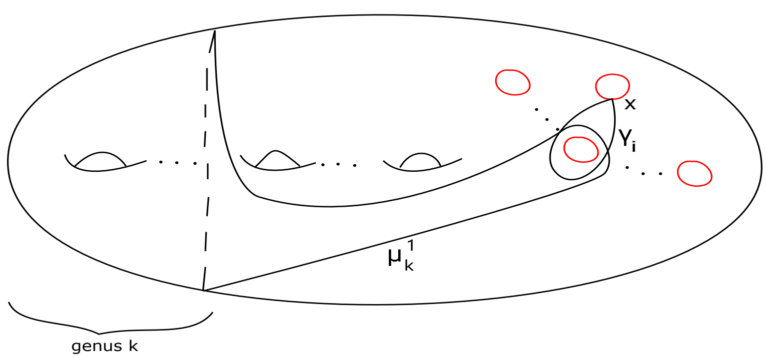

Figure 6: μ_k^1

$$\tau(T_{\mu_k^1})([\alpha_j]) = [\gamma_i] \wedge [\alpha_j], 1 \leq j \leq k$$
$$\tau(T_{\mu_k^1})([\beta_j]) = [\gamma_i] \wedge [\beta_j], 1 \leq j \leq k$$
$$\tau(T_{\mu_k^1})([c]) = 0, if\ c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_{b-1}\}$$

Third, for μ_k^2, supposing this curve separates the $(i+1)$-th and the $(j+1)$-th boundary components $(i < j)$ from all the other boundary components in the figure below, we will get:

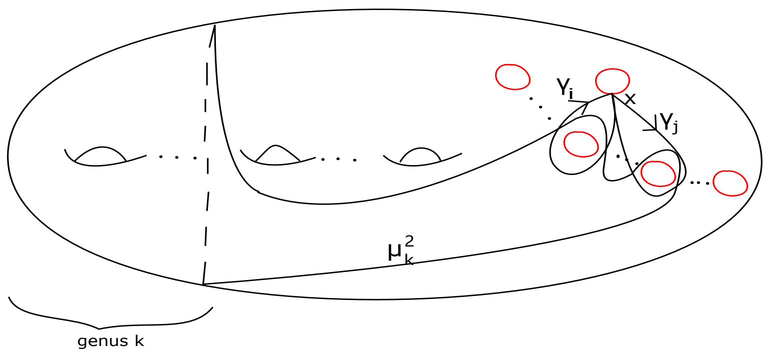

Figure 7: μ_k^2

$$\tau(T_{\mu_k^2})([\alpha_l]) = [\gamma_i] \wedge [\alpha_l] + [\gamma_j] \wedge [\alpha_l], 1 \leq l \leq k$$
$$\tau(T_{\mu_k^2})([\beta_l]) = [\gamma_i] \wedge [\beta_l] + [\gamma_j] \wedge [\beta_l], 1 \leq l \leq k$$
$$\tau(T_{\mu_k^2})([\gamma_i] = [\gamma_j] \wedge [\gamma_i] = -[\gamma_i] \wedge [\gamma_j]$$
$$\tau(T_{\mu_k^2})([\gamma_j] = [\gamma_i] \wedge [\gamma_j]$$
$$\tau(T_{\mu_k^2})([c]) = 0, if\ c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \hat{\gamma}_i, \cdots, \hat{\gamma}_j, \cdots, \gamma_{b-1}\}$$

Continuing this process, we can compute the image for $\mu_k^m (1 \leq m \leq b-1)$ which separates the (i_1+1)-th, $\ldots$, (i_m+1)-th boundary components from all the other boundariy components $(i_1 < i_2 < \cdots < i_m)$:

$$\tau(T_{\mu_k^m})([\alpha_j]) = \sum_{l=1}^{m} [\gamma_{i_l}] \wedge [\alpha_j], 1 \leq j \leq k$$
$$\tau(T_{\mu_k^m})([\beta_j]) = \sum_{l=1}^{m} [\gamma_{i_l}] \wedge [\beta_j], 1 \leq j \leq k$$
$$\tau(T_{\mu_k^m})([\gamma_{i_1}] = \sum_{l \neq 1} [\gamma_{i_l}] \wedge [\gamma_{i_1}]$$
$$\cdots$$
$$\tau(T_{\mu_k^m})([\gamma_{i_m}] = \sum_{l \neq m} [\gamma_{i_l} \wedge [\gamma_{i_m}]$$
$$\tau(T_{\mu_k^m})([c]) = 0, if\ c \in \{\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \hat{\gamma}_{i_1}, \cdots, \hat{\gamma}_{i_m}, \cdots, \gamma_{b-1}\}$$

(b) Bounding pair maps

Due to the definition of bounding pair maps, any bounding pair α,β must separate $\Sigma_{g,b}$ into a surface homeomorphic to $\Sigma_{k,2}(1 \le k \le g-1)$ and a surface homeomorphic to $\Sigma_{g-k-1,b+2}$ shown in the figure below.

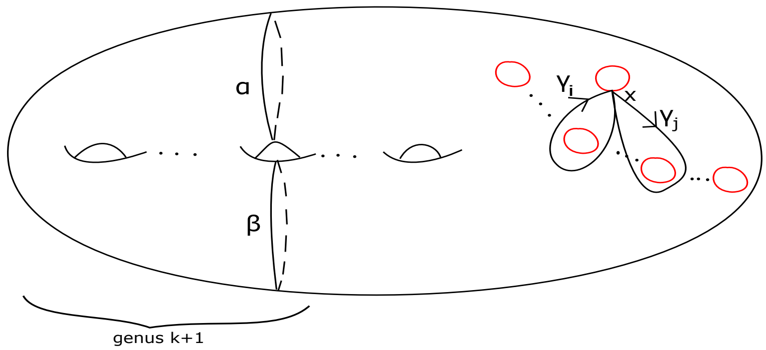

Figure 8: *a bounding pair in* $\Sigma_{g,b}$

Again, let $g_k = T_\alpha T_\beta^{-1}$, we can obtain similar results:

$$\tau(g_k)([\alpha_i]) = [\beta_{k+1}] \wedge [\alpha_i], 1 \le i \le k$$
$$\tau(g_k)([\beta_i]) = [\beta_{k+1}] \wedge [\beta_i], 1 \le i \le k$$
$$\tau(g_k)([\alpha_{k+1}]) = \sum_{i=1}^{k} [\alpha_i] \wedge [\beta_i]$$
$$\tau(g_k)([c]) = 0, if\ c \in \{\beta_{k+1}, \alpha_{k+2}, \beta_{k+2}, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_{b-1}\}$$

After observing the images of Dehn twists about separating curves and bounding pair maps in $\Sigma_{g,b}$ and letting $Mod(\Sigma_{g,b})$ act on them, we get the image for Johnson homomorphism for $\Sigma_{g,b}(b \ge 3)$ in the theorem as follows.

Theorem 5.1. *If we take the standard basis* $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_{b-1}\}$ *of* $\pi_1(\Sigma_{g,b}, x)(b \ge 3)$ *(see figure 4), the image of* $\tau : \mathscr{T}(\Sigma_{g,b}) \to Hom(H, \wedge^2 H)$ *in the matrix form is:*

$$\left\{ \begin{matrix} [\gamma_i]\wedge[\gamma_j] \\ [\gamma_1]\wedge[c_i] \\ \vdots \\ [\gamma_{b-1}]\wedge[c_i] \\ [c_i]\wedge[c_j] \end{matrix} \overset{\begin{matrix} \quad [\gamma_k] & \qquad [c_m] \end{matrix}}{\begin{pmatrix} E_{C_{b-1}^2 \times (b-1)} & F_{C_{b-1}^2 \times 2g} \\ 0 & \bigstar_1 \\ \vdots & \vdots \\ 0 & \bigstar_{b-1} \\ 0 & D \end{pmatrix}} \middle| D, E, F, \bigstar_i (1 \le i \le b-1) satisfy\ conditions\ beneath \right\} \tag{9}$$

(i) D can be any element in $\wedge^3 H'$ *with* $\wedge^3 H'$ *emdedded in* $Hom(H', \wedge^2 H')$.

(ii) F can be any matrix in $M_{C_{b-1}^2 \times 2g}(\mathbb{Z})$.

(iii)

$$(1 \leq i \leq b-1)\bigstar_i = \begin{pmatrix} A_i & B_i \\ C_i & A_i^t \end{pmatrix}, \forall A_i, B_i, C_i \in M_{g\times g}(\mathbb{Z}), B_i^t = -B_i, C_i^t = -C_i.$$

(iv) *For E,denote* $\langle [\gamma_k], [\gamma_i]\wedge[\gamma_j]\rangle$ *as the coefficient of* $[\gamma_i]\wedge[\gamma_j]$ *in* $\tau(f)([\gamma_k])$ *for a fixed element* $f \in \mathscr{T}(\Sigma_{g,b})$*, then we have:* $\langle [\gamma_i]+[\gamma_j], [\gamma_i]\wedge[\gamma_j]\rangle = \langle [\gamma_k], [\gamma_i]\wedge[\gamma_j]\rangle = 0, (i \neq j, k\neq i, k\neq j)$*. Then E could be any matrix in* $M_{C_{b-1}^2\times(b-1)}(\mathbb{Z})$ *satisfying the above condition.*

proof of theorem: Above all, the images we have calculated in **(a)** and **(b)** actually satisfy the form in (9). Then there are two things left for us to check. On the one hand, we should prove that after letting any mapping class act on the standard Dehn twists about separating curves and bounding pair maps, the images still lie in (9). On the other hand, we need to show that every element in (9) can be realized.

First, by reviewing Lemma 4.2, for any $\varphi \in Mod(\Sigma_{g,b})$, its corresponding matrix form could be:

$$P = \begin{pmatrix} \tilde{P}_{2g\times 2g} & 0 \\ v_1 & \\ \vdots & I_{b-1} \\ v_{b-1} & \end{pmatrix}, \forall \tilde{P} \in Sp(2g;\mathbb{Z}), \forall v_i \in M_{1\times 2g}(\mathbb{Z}), 1 \leq i \leq b-1$$

Then we examine carefully the affections of the action of φ.Since τ is linear, we only need to check different parts in the matrix (9) repectively as follows:

(I) Let $h_1 \in Hom(H, \wedge^2 H)$ correspond to the following matrix where E satisfies the condition (ii) in the proposition:

$$\begin{pmatrix} \mathrm{E}_{C_{b-1}^2\times(b-1)} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we have:

$$\varphi_*(h_1)([\gamma_i]) = \varphi_*(h_1(\varphi_*^{-1}([\gamma_i]))) = \varphi_*(h_1([\gamma_i])) = h_1([\gamma_i]), 1 \leq i \leq b-1$$

$$\varphi_*(h_1)([c_i]) = \varphi_*(h_1(\varphi_*^{-1}([c_i]))) = \varphi_*(h_1(\sum_{j=1}^{2g}\tilde{P}_{j,i}^{-1}[c_j] + \sum_{l=1}^{b-1} w_l^i[\gamma_l])) = \sum_{l=1}^{b-1} w_l^i h_1([\gamma_l]), 1 \leq i \leq 2g$$

Above computations just tell us $\varphi_*(h_1)$ is still in (9).

(II) Let $h_2 \in Hom(H, \wedge^2 H)$ correspond to the following matrix where F is arbitrary :

$$\begin{pmatrix} 0 & \mathrm{F}_{C_{b-1}^2\times 2g} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we have:

$$\varphi_*(h_2)([\gamma_i]) = \varphi_*(h_2(\varphi_*^{-1}([\gamma_i]))) = \varphi_*(h_2([\gamma_i])) = 0, 1 \le i \le b-1$$

$$\varphi_*(h_2)([c_i]) = \varphi_*(h_2(\varphi_*^{-1}([c_i]))) = \varphi_*(h_2(\sum_{j=1}^{2g} \tilde{P}_{j,i}^{-1}[c_j] + \sum_{l=1}^{b-1} w_l^i[\gamma_l]))$$

$$= \sum_{j=1}^{2g} \tilde{P}_{j,i}^{-1} \varphi_*(h_1([c_j])) = \sum_{j=1}^{2g} \tilde{P}_{j,i}^{-1} h_1([c_j]), 1 \le i \le 2g$$

This shows that $\varphi_*(h_2)$ lies in (9).

(III) Let $f_i \in Hom(H, \wedge^2 H)(1 \le i \le b-1)$ correspond to the following matrix where $\bigstar_i$ satisfies the condition (iii) in the proposition:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \bigstar_i \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we have:

$$\varphi_*(f_i)([\gamma_j]) = \varphi_*(f_i(\varphi_*^{-1}([\gamma_j]))) = \varphi_*(f_i([\gamma_j])) = 0, 1 \le j \le b-1$$

$$\varphi_*(f_i)([c_j]) = \varphi_*(f_i(\varphi_*^{-1}([c_j]))) = \varphi_*(f_i(\sum_{l=1}^{2g} \tilde{P}_{l,j}^{-1}[c_l] + \sum_{l=1}^{b-1} w_l^j[\gamma_l])) = \sum_{l=1}^{2g} \tilde{P}_{l,j}^{-1} \varphi_*(f_i([c_l]))$$

$$= \sum_{l=1}^{2g} \tilde{P}_{l,j}^{-1} \varphi_*(\sum_{k=1}^{2g} (\bigstar_i)_{k,l}[\gamma_i] \wedge [c_k]) = \sum_{l=1}^{2g}\sum_{k=1}^{2g} \tilde{P}_{l,j}^{-1} (\bigstar_i)_{k,l}[\gamma_i] \wedge \varphi_*([c_k])$$

$$= \sum_{l=1}^{2g}\sum_{k=1}^{2g} \tilde{P}_{l,j}^{-1} (\bigstar_i)_{k,l}[\gamma_i] \wedge (\sum_{m=1}^{2g} \tilde{P}_{m,k}[c_m] + \sum_{m=1}^{b-1} v_m^k[\gamma_m])$$

$$= \sum_{l=1}^{2g}\sum_{k=1}^{2g}\sum_{m=1}^{2g} \tilde{P}_{l,j}^{-1} (\bigstar_i)_{k,l}\tilde{P}_{m,k}[\gamma_i] \wedge [c_m] + \sum_{l=1}^{2g}\sum_{k=1}^{2g}\sum_{m=1}^{b-1} \tilde{P}_{l,j}^{-1} (\bigstar_i)_{k,l} v_m^k[\gamma_i] \wedge [\gamma_m], 1 \le j \le 2g$$

Then the matrix of $\varphi_*(f_i)$ is:

$$\begin{pmatrix} 0 & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \tilde{P}\bigstar_i\tilde{P}^{-1} \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And from Proposition 4.3 we know that $\tilde{P}\bigstar_i\tilde{P}^{-1}$ still satisfies condition (iii), thus $\varphi_*(f_i)$ lies in (9).

(IV) Let g_k be the bounding pair map in (**b**), and we know the matrix corresponding to φ^{-1} is:

$$P^{-1}=\begin{pmatrix}\tilde{P}^{-1} & 0\\ w_1 & \\ \vdots & I_{b-1}\\ w_{b-1} & \end{pmatrix}$$

Then we have:

$$\varphi_*(g_k)([\gamma_i])=\varphi_*(g_k(\varphi_*^{-1}([\gamma_i])))=\varphi_*(g_k([\gamma_i]))=0, 1\le i\le b-1$$

$$\varphi_*(g_k)([c_i])=\varphi_*(g_k(\varphi_*^{-1}([c_i])))=\varphi_*(g_k(\sum_{j=1}^{2g}(\tilde{P}^{-1})_{j,i}[c_j]+\sum_{j=1}^{b-1}w_j^i[\gamma_j]))$$

$$=\varphi_*(g_k(\sum_{j=1}^{k}\tilde{P}_{j,i}^{-1}[\alpha_j]))+\varphi_*(g_k(\tilde{P}_{k+1,i}^{-1}[\alpha_{k+1}]))+\varphi_*(g_k(\sum_{j=g+1}^{g+k}\tilde{P}_{j,i}^{-1}[\beta_{j-g}]))$$

$$=\sum_{j=1}^{k}\tilde{P}_{j,i}^{-1}\varphi_*([\beta_{k+1}]\wedge[\alpha_j])+\tilde{P}_{k+1,i}^{-1}\varphi_*(\sum_{l=1}^{k}[\beta_l]\wedge[\alpha_l])+\sum_{j=g+1}^{g+k}\tilde{P}_{j,i}^{-1}\varphi_*([\beta_{k+1}]\wedge[\beta_{j-g}])$$

$$=\sum_{j=1}^{k}\tilde{P}_{j,i}^{-1}(\sum_{m=1}^{b-1}v_m^{g+k+1}[\gamma_m]+\sum_{m=1}^{2g}\tilde{P}_{m,g+k+1}[c_m])\wedge(\sum_{n=1}^{b-1}v_n^j[\gamma_n]+\sum_{n=1}^{2g}\tilde{P}_{n,j}[c_n])$$

$$+\tilde{P}_{k+1,i}^{-1}\sum_{l=1}^{k}(\sum_{m=1}^{b-1}v_m^l[\gamma_m]+\sum_{m=1}^{2g}\tilde{P}_{m,l}[c_m])\wedge(\sum_{n=1}^{b-1}v_n^{g+l}[\gamma_n]+\sum_{n=1}^{2g}\tilde{P}_{n,g+l}[c_n])$$

$$+\sum_{j=g+1}^{g+k}\tilde{P}_{j,i}^{-1}(\sum_{m=1}^{b-1}v_m^{g+k+1}[\gamma_m]+\sum_{m=1}^{2g}\tilde{P}_{m,g+k+1}[c_m])\wedge(\sum_{n=1}^{b-1}v_n^j[\gamma_n]+\sum_{n=1}^{2g}\tilde{P}_{n,j}[c_n]), 1\le i\le 2g$$

The part of $[\gamma_m]\wedge[\gamma_n]$ can be neglected since its coefficient can be arbitrary; the part of $[c_m]\wedge[c_n]$ is just the image of $\tilde{\varphi}_*(\tilde{f}_k)$ where $\tilde{\varphi}\in Mod(\Sigma_{g,1})$ corresponds to the matrix $\tilde{P}\in M_{2g\times 2g}(\mathbb{Z})$ and $\tilde{f}_k$ is the image of f_k under $\mathscr{T}(\Sigma_{g,b})\to\mathscr{T}(\Sigma_{g,b-1})\to\cdots\to\mathscr{T}(\Sigma_{g,1})$. And Proposition 4.1 tells us that this part lies in $\wedge^3H'$; the last part is a multiple of $[\gamma_m]\wedge[c_n]$. Denote $Q_{i,n}^m$ as the coefficient of $[\gamma_m]\wedge[c_n]$ in $\varphi_*(f_k(\varphi_*^{-1}([c_i])))$, then we have:

$$\begin{aligned}Q_{i,n}^m=&\sum_{j=1}^{k}v_m^{g+k+1}\tilde{P}_{j,i}^{-1}\tilde{P}_{n,j}+\sum_{j=g+1}^{g+k}v_m^{g+k+1}\tilde{P}_{j,i}^{-1}\tilde{P}_{n,j}\\&-\sum_{j=1}^{k}v_m^j\tilde{P}_{j,i}^{-1}\tilde{P}_{n,g+k+1}+\sum_{j=1}^{k}v_m^j\tilde{P}_{k+1,i}^{-1}\tilde{P}_{n,g+j}\\&-\sum_{j=1}^{k}v_m^{g+j}\tilde{P}_{k+1,i}^{-1}\tilde{P}_{n,j}-\sum_{j=g+1}^{g+k}v_m^j\tilde{P}_{j,i}^{-1}\tilde{P}_{n,g+k+1}\end{aligned}$$

Then in the same way in section 4, we can prove that for each fixed $m(1\le m\le b-1)$, the correponding matrix satisfies condition (iii).

So far, through (I)(II)(III)(IV), we've proven that the images obtained by letting any mapping class act on Dehn twists about separating curves and bounding pair maps are truly in (9), namely $\tau(\mathscr{T}(\Sigma_{g,b}))$ is a subset of (9). Then we continue to prove that any element in 9 can be realized as an image.

Firstly, for the separating curve μ_k^1 in (**a**) that separates the $(i+1)$-th boundary component from others ($1 \le i \le b-1$), and for $\varphi \in Mod(\Sigma_{g,b})$ whose corresponding matrix is:

$$P = \begin{pmatrix} \tilde{P} & 0 \\ 0 & \\ \vdots & \\ v_i & I_{b-1} \\ \vdots & \\ 0 & \end{pmatrix}, \forall \tilde{P} \in Sp(2g;\mathbb{Z}), \forall v_i \in M_{1\times 2g}(\mathbb{Z})$$

Then similar to the process for $b=2$, we find that $\varphi_*(\mu_k^1)$ (for fixed i) spans:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \bigstar_i \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \bigstar_i = \begin{pmatrix} A_i & B_i \\ C_i & A_i^t \end{pmatrix}, \forall A_i, B_i, C_i \in M_{g\times g}(\mathbb{Z}), B_i^t = -B_i, C_i^t = -C_i.$$

Secondly, for the separating curve μ_0^2 in ((a)) that separates the $(i+1)$-th and the $(j+1)$-th boundary components $(i<j)$ from all other boundary components and separates $\Sigma_{g,b}$ into a surface homeomorphic to $\Sigma_{0,3}$ and the other surface homeomorphic to $\Sigma_{g,b-1}$. Then we have:

$$\tau(T_{\mu_0^2})([\gamma_i]) = -[\gamma_i]\wedge[\gamma_j]$$
$$\tau(T_{\mu_0^2})([\gamma_j] = [\gamma_i]\wedge[\gamma_j]$$
$$\tau(T_{\mu_0^2})([c]) = 0, if\ c \in \{\alpha_1,\beta_1,\cdots,\alpha_g,\beta_g,\gamma_1,\cdots,\hat{\gamma}_i,\cdots,\hat{\gamma}_j,\cdots,\gamma_{b-1}\}$$

Since here i and j could be any intergers in $\{1,2,\ldots,b-1\}$ with $(i<j)$,then it will span:

$$\begin{pmatrix} \mathrm{E}_{C_{b-1}^2\times(b-1)} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{any E satisfies condition(iv)}$$

Thirdly, let any $\varphi \in Mod(\Sigma_{g,b})$ act on $T_{\mu_0^2}$ above, then:

$$\varphi_*(T_{\mu_0^2})([\gamma_i]) = \varphi_*(T_{\mu_0^2}(\varphi_*^{-1}([\gamma_i]))) = \varphi_*(T_{\mu_0^2}([\gamma_i])) = \varphi_*(-[\gamma_i]\wedge[\gamma_j]) = -[\gamma_i]\wedge[\gamma_j]$$
$$\varphi_*(T_{\mu_0^2})([\gamma_j]) = \varphi_*(T_{\mu_0^2}(\varphi_*^{-1}([\gamma_j]))) = \varphi_*(T_{\mu_0^2}([\gamma_j])) = \varphi_*([\gamma_i]\wedge[\gamma_j]) = [\gamma_i]\wedge[\gamma_j]$$
$$\varphi_*(T_{\mu_0^2})([\gamma_l]) = \varphi_*(T_{\mu_0^2}(\varphi_*^{-1}([\gamma_l]))) = \varphi_*(T_{\mu_0^2}([\gamma_l])) = 0, 1 \le l \le b-1, l \ne i, l \ne j$$
$$\varphi_*(T_{\mu_0^2})([c_k]) = \varphi_*(T_{\mu_0^2}(\varphi_*^{-1}([c_k]))) = \varphi_*(T_{\mu_0^2}(\sum_{j=1}^{2g}\tilde{P}_{j,k}^{-1}[c_j] + \sum_{l=1}^{b-1} w_l^k[\gamma_l]))$$
$$= \varphi_*(w_i^k T_{\mu_0^2}([\gamma_i]) + w_j^k T_{\mu_0^2}([\gamma_j])) = (w_j^k - w_i^k)[\gamma_i]\wedge[\gamma_j], 1 \le l \le 2g$$

Then the matrix of $\varphi_*(T_{\mu_0^2}) - T_{\mu_0^2}$ becomes:

$$\begin{pmatrix} & 0 \\ & \vdots \\ 0 & w_j - w_i \\ & \vdots \\ & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Here w_i and w_j are arbitrary vectors since $v_l (1 \le l \le b-1)$ is from Lemma 4.2, thus it will span:

$$\begin{pmatrix} 0 & F_{C_{b-1}^2 \times 2g} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{F is arbitrary}$$

Fourthly and lastly, since $\mathscr{T}(\Sigma_{g,b}) \to \mathscr{T}(\Sigma_{g,b-1}) \to \cdots \to \mathscr{T}(\Sigma_{g,1})$ are all surjective, so D in the proposition could be any element in $\wedge^3 H'$. Thus finally we can say any element in (9) can be realized, then our proof is done. □

Remark 5.2. So far we have computed out the images of Johnson homomorphism for all compact surfaces, but especially we describe the images by choosing a special basis of $\pi_1(\Sigma_{g,b})$ and taking the matrix form. We are still looking forward to finding a more intrinsic description.

6 A special case for surfaces of infinte type

In this section, we try to compute the image of Johnson homomorphism for a special surface of infinite topological type in the following Figure 9. Before that, we have to know some basics of surfaces of infinite type.

A surface Σ of infinite type is a connected oriented surface whose fundamental group is not finitely generated. And according to [10], the homoeomorphism type of Σ of infinite type is determined by the tuple:

$$(g(\Sigma), b(\Sigma), Ends(\Sigma), Ends_g(\Sigma))$$

where $g(\Sigma)$ is the genus of Σ; $b(\Sigma)$ is the number of boundary components of Σ; $Ends(\Sigma)$ is the space of ends of Σ which is defined as $Ends(\Sigma) = \varprojlim \pi_0(\Sigma \backslash K)$ where the inverse limit is taken over the set of compact subsets $K \subset \Sigma$ directed with repect to inclusion; $Ends_g(\Sigma)$ is the subset of $Ends(\Sigma)$ consisting of ends accumulated by genus, where we say $e \in Ends(\Sigma)$ is accumulated by genus if every neighborhood of e has infinite genus. Then our surface in figure 10 correponds to the tuple: $(+\infty, 1, \{e\}, \{e\})$.
For a surface Σ, we could also define its mapping class group, which is usually called a big mapping class group:

$$Mod(\Sigma) = Homeo(\Sigma, \partial\Sigma) \backslash Homeo_0(\Sigma, \partial\Sigma)$$

where $Homeo(\Sigma, \partial\Sigma)$ is the group of self homeomorphisms of σ which fixes $\partial\Sigma$ pointwise, and $Homeo_0(\Sigma, \partial\Sigma)$ is the connected component of the identity in $Homeo(\Sigma, \partial\Sigma)$. As for the topology, after equipping $Homeo(\Sigma, \partial\Sigma)$ with the compact-open topology, $Mod(\Sigma)$ is then equipped with the quotient topology naturally. We denote this topology as τ_q and $PMod(\Sigma)$ is the pure mapping class group whose elements act trivially on $Ends(\Sigma)$ with the subspace topology. The following is a part of Proposition 6.2 in [7]:

Proposition 6.1 (P.P&N.G.V)**.** *If Σ has at most one end accumulated by genus, the set of Dehn twists topologically generates $(PMod(\Sigma), \tau_q)$.*

Notice there's a difference between generating and topologically generating.
We care about the Torelli group of Σ which is also difined as:

$$\mathscr{T}(\Sigma) := \text{Ker of } (Mod(\Sigma) \to Aut(H_1(\Sigma;\mathbb{Z})))$$

Then $\mathscr{T}(\Sigma) < PMod(\Sigma)$ is equipped with the subspace topology. And we denote by $\mathscr{T}_c(\Sigma)$ the subgroup of $\mathscr{T}(\Sigma)$ consisting of elements with compact surports, and let $\overline{\mathscr{T}_c(\Sigma)}$ be the closure of $\mathscr{T}(\Sigma)$ in $PMod(\Sigma)$. Then there follows an important theorem from [3].

Theorem 6.2 (J.A,T.G,$\cdots$)**.** *For any connectd oriented surface Σ of infinite type, we have $\mathscr{T}(\Sigma) = \overline{\mathscr{T}_c(\Sigma)}$.*

In the following text, we care about surfaces of infinite type with $b(\Sigma) \geq 1$ so that Σ has at least one fixed point x. And through a similar procedure in section 3, we could define the Johnson homomorphism for Σ of infinite type (the only difference from finite cases is that $\pi_1(\Sigma, x)$ is now a free group with infinite generators):

$$\tau : \mathscr{T}(\Sigma) \to Hom(H_1(\Sigma;\mathbb{Z}), \wedge^2 H_1(\Sigma;\mathbb{Z}))$$

Lemma 6.3. *If $Hom(H_1(\Sigma;\mathbb{Z}), \wedge^2 H_1(\Sigma;\mathbb{Z}))$ is equipped with the compact-open topology, then the Johnson homomorphism τ is continuous.*

proof of lemma: The Johnson homomorphism can be separated into two steps:

$$\tau_1 : \mathscr{T}(\Sigma) \to Aut(\pi_1(\Sigma,x))$$
$$\tau_2 : Aut(\pi_1(\Sigma,x)) \to Hom(H_1(\Sigma;\mathbb{Z}), \wedge^2 H_1(\Sigma;\mathbb{Z}))$$
$$f \mapsto (y \to f(y)y^{-1})$$

Here $\tau = \tau_2 \circ \tau_1$ and the topology of $Aut(\pi_1(\Sigma,x))$ is also the compact-open topology, so we only need to prove τ_1 and τ_2 are continuous.

For τ_1, suppose g_n converges to g in $\mathscr{T}(\Sigma)$. For any $y \in \pi_1(\Sigma,x)$ that can be represented by a simple closed curve, $\tau_1(g)(y)$ also can be represented by a simple closed curve. Then we take $U \subset \Sigma$ to be the curve corresponding to y, and take $V \subset \Sigma$ to be the regular neighborhood of the curve corresponding to $\tau_1(g)(y)$. Since it's evident that U is compact and V is open, there exists an integer N such that $\forall n > N$, g_n maps U into V, so $\tau_1(g_n)$ maps y into $\tau_1(g)(y)$, namely $\tau_1(g_n)$ converges to $\tau_1(g)$ in $Aut(\pi_1(\Sigma,x))$. Thus τ_1 is continuous.

For τ_2, if f_n converges to f in $Aut(\pi(\Sigma,x))$, it means that for any $y \in \pi_1(\Sigma,x)$, there exists a large enough integer N such that $\forall n > N$, we have $f_n(y) = f(y)$, which directly shows that $\tau_2(f_n)$ converges to $\tau_2(f)$. So τ_2 is continuous.

From above, we prove that τ is a continuous map. □

Use the above lemma, the following corollary is an easy exercise.

Corollary 6.4. *$\tau(\mathscr{T}(\Sigma)) \subset \overline{\bigcup_{\Sigma_c \subset \Sigma} \tau(\mathscr{T}(\Sigma_c))}$, where Σ_c is any compact subsurface of Σ.*

proof of corollary: First, from Theorem 6.3, we have: $\mathscr{T}(\Sigma) = \overline{\bigcup_{\Sigma_c \subset \Sigma} \mathscr{T}(\Sigma_c)}$.

Since τ is continuous due to Lemma 6.4, we have:

$$\tau(\mathscr{T}(\Sigma)) = \tau(\overline{\bigcup_{\Sigma_c \subset \Sigma} \mathscr{T}(\Sigma_c)}) \subset \overline{\tau(\bigcup_{\Sigma_c \subset \Sigma} \mathscr{T}(\Sigma_c))} = \overline{\bigcup_{\Sigma_c \subset \Sigma} \tau(\mathscr{T}(\Sigma_c))}$$

□

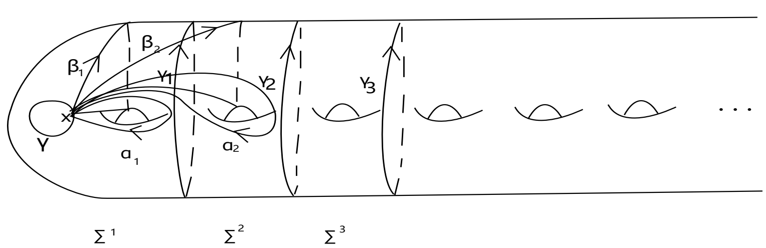

Figure 9: *a special surface of infinite type*

Now, for our special surface Σ in figure 10. Denote respectively by $\Sigma^1, \Sigma^2, \cdots, \Sigma^n$ the surfaces obtained by cutting off the first $1, 2, \cdots, n$ genus, and denote the boundary of Σ^n as γ_n. We can see that Σ^n is homeomorphic to $\Sigma_{n,2}$, and Corollary 6.4 tells us that $\tau(\mathscr{T}(\Sigma)) = \overline{\bigcup_{n=1}^{+\infty} \tau(\mathscr{T}(\Sigma^n))}$. Now take a standard basis $\{\gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots\}$ of $\pi_1(\Sigma,x)$. Then $\{\gamma, \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n\}$ is a basis of

$\pi_1(\Sigma^n,x)$, and in $H_1(\Sigma)$ we have $[\gamma]=[\gamma_1]=[\gamma_2]=\cdots=[\gamma_n]=\cdots$. Also, the basis of $H_1(\Sigma;\mathbb{Z})$ is $\{[\gamma],[\alpha_1],[\beta_1],[\alpha_2],[\beta_2],\cdots\}$.

Using these facts, we could calculate the image of Johnson homomorphism for Σ as follows. But since the generators of $\pi_1(\Sigma,x)$ are infinite, we aviod using matrix to describe the images, but use conditions satisefied by all coefficients instead.

Theorem 6.5. *Σ is the surface in figure 10, $\{\gamma,\alpha_1,\beta_1,\alpha_2,\beta_2,\cdots\}$ is a standard basis of $\pi_1(\Sigma,x)$. Then $\tau(\mathscr{T}(\Sigma))$ is the subset of $Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2H_1(\Sigma;\mathbb{Z}))$ satisfying the following conditions:*

(i) Any $f\in\tau(\mathscr{T}(\Sigma))$, $f([\gamma])=0$.

(ii) For $f\in Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2H_1(\Sigma;\mathbb{Z}))$, denote by $A_{j,i}$ the coefficient of $[\gamma]\wedge[\alpha_j]$ in $f([\alpha_i])$; denote by $B_{j,i}$ the coefficient of $[\gamma]\wedge[\alpha_j]$ in $f([\beta_i])$; denote by $C_{j,i}$ the coefficient of $[\gamma]\wedge[\beta_j]$ in $f([\alpha_i])$; denote by $D_{j,i}$ the coefficient of $[\gamma]\wedge[\beta_j]$ in $f([\beta_i])$. Then $f\in\tau(\mathscr{T}(\Sigma))$ satisfies:

$$A_{j,i}=D_{i,j};B_{j,i}=-B_{i,j};C_{j,i}=-C_{i,j}(\forall i\geq 1,j\geq 1)$$

as well as for any fixed integer i, there are finite nonzero numbers in $\{A_{i,j},B_{i,j},C_{i,j},D_{i,j},A_{j,i},B_{j,i},C_{j,i},D_{j,i}|\forall j\in\mathbb{N}\}$.

(iii) For convenience, denote $c_1=\alpha_1,c_2=\beta_1,c_3=\alpha_2,c_4=\beta_2,\cdots$. In $H_1(\Sigma^n;\mathbb{Z})\cong H_1(\Sigma^n;\mathbb{Z})^*$, if we denote $\hat{[c_k]}$ as the dual of $[c_k]$, we know that $[\alpha_k]$ maps to $\hat{[\beta_k]}$ and $[\beta_k]$ maps to $-\hat{[\alpha_k]}$ in this isomorphism. Then $f\in\tau(\mathscr{T}(\Sigma))$ satisfies: If the coefficient of $[c_j]\wedge[\gamma_k]$ in $f(\hat{[c_i]})$ is m, the coefficient of $[c_k]\wedge[\gamma_i]$ in $f(\hat{[c_j]})$ must be m and the coefficient of $[c_i]\wedge[\gamma_j]$ in $f(\hat{[c_k]})$ must be m; Also for any fixed integer i, there are finitely many nonzero terms of $[c_j]\wedge[c_k]$ in $f([c_i])$, and for any fixed integers $j<k$, there are finitely many integers i such that $f([c_i])$ contains nonzero $[c_j]\wedge[c_k]$.

proof of theorem: On the one hand, we need to prove that for any $g\in\mathscr{T}(\Sigma)$, $\tau(g)$ satisfies these three conditions.

First, it's easy to see that $\tau(g)([\gamma])=0$ since $g_*(\gamma)=\gamma$ in $\pi_1(\Sigma)$.

Second, we have shown in section 4, any element in $\tau(\mathscr{T}(\Sigma_n)$ satisfies the condition:

$$A_{j,i}=D_{i,j};B_{j,i}=-B_{i,j};C_{j,i}=-C_{i,j}(\forall i\geq 1,j\geq 1)$$

since here $A=D^t,B^t=-B,C^t=-C$ for the first n rows and first n columns, while the other entries are zero. And since $X=\{f\in Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2H_1(\Sigma;\mathbb{Z}))|A_{j,i}=D_{i,j};B_{j,i}=-B_{i,j};C_{j,i}=-C_{i,j}\}$ is a closed subset of $Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2H_1(\Sigma;\mathbb{Z}))$, and we have learned that $\forall n\in\mathbb{N},\tau(\mathscr{T}(\Sigma^n))\subset X$. Knowing that $\tau(\mathscr{T}(\Sigma))\subset\overline{\bigcup_{n=1}^{+\infty}\tau(\mathscr{T}(\Sigma^n))}$, we see $\tau(\mathscr{T}(\Sigma))\subset X$. Besides, for any fixed integer i, it's natural that there are only finite nonzero numbers in $\{A_{j,i},B_{j,i},C_{j,i},D_{j,i}|\forall j\in\mathbb{Z}\}$ due to the definition of τ. Using this fact and the properties satisfies by $A_{i,j},B_{i,j},C_{i,j},D_{i,j}$, there are also finitely many nonzero numbers in $\{A_{i,j},B_{i,j},C_{i,j},D_{i,j}|\forall j\in\mathbb{Z}\}$.

Third, we know that for any $f\in\tau(\mathscr{T}(\Sigma^n))$, if $f(\hat{[c_i]})$ contains a $m[c_j]\wedge[c_k]$, since this part in its image should lie in $\wedge^3H_1(\Sigma_{n,1};\mathbb{Z})$, there must be a $m[c_i]\wedge[c_j]\wedge[c_k]\in\wedge^3H_1(\Sigma_{n,1};\mathbb{Z})$ where $\wedge^3H_1(\Sigma_{n,1};\mathbb{Z})$ embeds in $Hom(H_1(\Sigma_{n,1};\mathbb{Z}),\wedge^2H_1(\Sigma_{n,1};\mathbb{Z}))$ by mapping $a\wedge b\wedge c$ to $\hat{a}\otimes(b\wedge c)+\hat{b}\otimes(c\wedge a)+\hat{c}\otimes(a\wedge b)$, so there must be a $m[c_k]\wedge[c_i]$ in $f(\hat{[c_j]})$ and a $m[c_i]\wedge[c_j]$ in $f(\hat{[c_k]})$. So any element in $\tau(\mathscr{T}(\Sigma^n))$ satisfies the first part of condition (iii), and similarly elements satisfying this condition make up a closed subset of $Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2 H_1(\Sigma;\mathbb{Z}))$, thus $\tau(\mathscr{T}(\Sigma)) \subset \overline{\bigcup_{n=1}^{+\infty} \tau(\mathscr{T}(\Sigma^n))}$ also satisfies this condition. Also, it's natural for a fixed integer i, there are finitely many nonzero terms of $[c_j]\wedge[c_k]$ in $f([c_i])$. Due to the fact that there must be finitely many nonzero $[c_i]\wedge[c_j]\wedge[c_k]$ in the image, we see that for any fixed integers $j<k$, there are finitely many integers i such that $f([c_i])$ contains nonzero $[c_j]\wedge[c_k]$.

On the other hand, we have to show that any $f \in Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2 H_1(\Sigma;\mathbb{Z}))$ satisfying the three conditions in the proposition could be realized as the image of some $g \in \mathscr{T}(\Sigma)$. We construct g in the following way:

First construct $f_1 \in Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2 H_1(\Sigma;\mathbb{Z}))$, if we denote by $(A_{i,j})_f,(B_{i,j})_f,(C_{i,j})_f,(D_{i,j})_f$ the coefficients for f with the meaning the same as condition (ii), and $(A_{i,j})_{f_1},(B_{i,j})_{f_1},(C_{i,j})_{f_1},(D_{i,j})_{f_1}$ are coefficients for f_1. We define:

$$\forall i\in\mathbb{N},(A_{1,i})_{f_1}=(A_{1,i})_f,(A_{i,1})_{f_1}=(A_{i,1})_f$$
$$\forall i\in\mathbb{N},(B_{1,i})_{f_1}=(B_{1,i})_f,(B_{i,1})_{f_1}=(B_{i,1})_f$$
$$\forall i\in\mathbb{N},(C_{1,i})_{f_1}=(C_{1,i})_f,(C_{i,1})_{f_1}=(C_{i,1})_f$$
$$\forall i\in\mathbb{N},(D_{1,i})_{f_1}=(D_{1,i})_f,(D_{i,1})_{f_1}=(D_{i,1})_f$$
$$\forall j\neq 1,k\neq 1,(A_{j,k})_{f_1}=(B_{j,k})_{f_1}=(C_{j,k})_{f_1}=(D_{j,k})_{f_1}=0$$

There are finitely many nonzero numbers in $\{(A_{1,i})_f,(A_{i,1})_f,(B_{1,i})_f,(B_{i,1})_f,(C_{1,i})_f,(C_{i,1})_f,(D_{1,i})_f,(D_{i,1})_f|\forall i\in\mathbb{N}\}$ since f satisfies condition(ii), thus there are finite nonzero numbers in $\{(A_{i,j})_{f_1},(B_{i,j})_{f_1},(C_{i,j})_{f_1},(D_{i,j})_{f_1}|\forall i,j\in\mathbb{N}\}$. Also, supposing the part of f in $\wedge^3 H(\Sigma;\mathbb{Z})$ containing α_1 and β_1 is $\sum_{i=1}^{t} a_i[\alpha_1]\wedge[\beta_1]\wedge[c_{k_i}]+\sum_{i,j} b_{i,j}[\alpha_1]\wedge[c_{m_i}]\wedge[c_{n,j}]+\sum_{i,j} c_{i,j}[\beta_1]\wedge[c_{r_i}]\wedge[c_{s_j}]$ where all k_i,m_i,n_j,r_i,s_j are not 1 or 2 and there are finite nonzero $a_i,b_{i,j},c_{i,j}$. And we just define the part of f_1 in $\wedge^3 H(\Sigma;\mathbb{Z})$ with this term. That's to say:

$$f_1([\alpha_1])=\sum_i (A_{i,1})_f[\gamma]\wedge[\alpha_i]+\sum_i (C_{i,1})_f[\gamma]\wedge[\beta_i]+\sum_i a_i[\alpha_1]\wedge[c_{k_i}]+\sum_{i,j} c_{i,j}[c_{s_j}]\wedge[c_{r_i}]$$
$$f_1([\beta_1])=\sum_i (B_{i,1})_f[\gamma]\wedge[\alpha_i]+\sum_i (D_{i,1})_f[\gamma]\wedge[\beta_i]+\sum_i a_i[\beta_1]\wedge[c_{k_i}]+\sum_{i,j} b_{i,j}[c_{m_i}]\wedge[c_{n_j}]$$
$$\text{For } l\geq 1, f_1([c_{2l+1}])=(A_{1,l+1})_f[\gamma]\wedge[\alpha_1]+(C_{1,l+1})_f[\gamma]\wedge[\beta_1]+\sum_i \delta_{2l+2}^{k_i}a_i[\beta_1]\wedge[\alpha_1]+$$
$$\sum_{i,j}(\delta_{2l+2}^{m_i}b_{i,j}[\alpha_1]\wedge[c_{n,j}]+\delta_{2l+2}^{n_j}b_{i,j}[c_{m,j}]\wedge[\alpha_1])+\sum_{i,j}(\delta_{2l+2}^{r_i}c_{i,j}[\beta_1]\wedge[c_{s,j}]+\delta_{2l+2}^{s_j}c_{i,j}[c_{r,j}]\wedge[\beta_1])$$
$$\text{For } l\geq 2, f_1([c_{2l}])=(B_{1,l})_f[\gamma]\wedge[\alpha_1]+(D_{1,l})_f[\gamma]\wedge[\beta_1]+\sum_i \delta_{2l-1}^{k_i}a_i[\alpha_1]\wedge[\beta_1]+$$
$$\sum_{i,j}(\delta_{2l-1}^{m_i}b_{i,j}[c_{n,j}]\wedge[\alpha_1]+\delta_{2l-1}^{n_j}b_{i,j}[\alpha_1])\wedge[c_{m,j}]+\sum_{i,j}(\delta_{2l-1}^{r_i}c_{i,j}[c_{s,j}]\wedge[\beta_1]+\delta_{2l-1}^{s_j}c_{i,j}[\beta_1]\wedge[c_{r,j}])$$

So we see that f_1 can be retricted in a finite matrix and it satisfies the form in Proposition 4.4, thus f_1 is the image of some $g_1\in\mathscr{T}(\Sigma^{n_1})$ for some $n_1\in\mathbb{N}$.

Then we construct $f_2\in Hom(H_1(\Sigma;\mathbb{Z}),\wedge^2 H_1(\Sigma;\mathbb{Z}))$ as follows:

$$\forall i \geq 2, (A_{2,i})_{f_2} = (A_{2,i})_f, (A_{i,2})_{f_2} = (A_{i,2})_f$$
$$\forall i \geq 2, (B_{2,i})_{f_2} = (B_{2,i})_f, (B_{i,2})_{f_2} = (B_{i,2})_f$$
$$\forall i \geq 2, (C_{2,i})_{f_2} = (C_{2,i})_f, (C_{i,2})_{f_2} = (C_{i,2})_f$$
$$\forall i \geq 2, (D_{2,i})_{f_2} = (D_{2,i})_f, (D_{i,2})_{f_2} = (D_{i,2})_f$$
$$\text{Foe all other } (j,k), (A_{j,k})_{f_2} = (B_{j,k})_{f_2} = (C_{j,k})_{f_2} = (D_{j,k})_{f_2} = 0$$

And likely we assign the part of f in $\wedge^3 H(\Sigma;\mathbb{Z})$ containing α_2 and β_2 but not containing α_1 and β_1 to f_2. Thus we can find $g_2 \in \mathscr{T}(\Sigma^{n_2})$ for some $n_2 \in \mathbb{N}$ such that $\tau(g_2) = f_2$.

Inductively, we could define $f_k \in Hom(H_1(\Sigma;\mathbb{Z}), \wedge^2 H_1(\Sigma;\mathbb{Z}))$ and there is $g_k \in \mathscr{T}(\Sigma^{n_k})$ for some $n_k \in \mathbb{N}$ such that $\tau(g_k) = f_k$.

From our constuctions, we can observe that f_2 preseves $[\alpha_1]$ and $[\beta_1]$ and does not map other curves to them, so actually we could realize $g_2 \in \mathscr{T}(\Sigma^{n_2} \backslash \Sigma^1)$. Also, for $\forall k \in \mathbb{N}$, f_k preserves $[\alpha_1], [\beta_1], \cdots, [\alpha_{k-1}], [\beta_{k-1}]$ and no other curves are mapped to them, so g_k can be realized in $\mathscr{T}(\Sigma^{n_k} \backslash \Sigma^{k-1})$.

Finally, we define $\tilde{g_n} = g_1 \cdot g_2 \cdots g_n$,and define $\tilde{f_n} = \sum_{i=1}^n f_i$. First we see that $g_2 g_3 \cdots g_n$ acts trivially on Σ^1, and $f_{k+1} f_{k+2} \cdots f_n$ acts trivially on Σ^k. This means that for any $\Sigma^k \subset \Sigma$, there exists an integer $N = k$, for any $m, n > N$, $\tilde{g_m}|_{\Sigma^k} = \tilde{g_n}|_{\Sigma^k}$, which is exactly what is meant by $\tilde{g_n}$ converges in the compact-open topology. Thus we can assume $\tilde{g_n}$ converges to $g \in \mathscr{T}(\Sigma)$. And from the constructions, we know that $\tilde{f_1}([\alpha_1]) = f([\alpha_1]), \tilde{f_1}([\beta_1]) = f([\beta_1])$, moreover, $\tilde{f_k}([\alpha_i]) = f([\alpha_i]), \tilde{f_k}([\beta_i]) = f([\beta_i]) (\forall 1 \leq i \leq k)$. So $\tilde{f_n}$ converges to f in $Hom(H_1(\Sigma;\mathbb{Z}), \wedge^2 H_1(\Sigma;\mathbb{Z}))$ equipped with the compact-open topology. Last, since $\tau(\tilde{g_n}) = \tau(g_1 g_2 \cdots g_n) = f_1 + f_2 + \cdots + f_n = \tilde{f_n}$ and τ is continuous, we get $\tau(g) = f$ after taking limits on both sides. Thus f is truly in the image of Johnson homomorphism.

Now our images are completely computed out and well proved. □

For surfaces of infinite type, we just compute a simplest one while other cases could be a lot different and more complex. It's still an interesting problem how to compute the images of Johnson homomorphism for more kinds of surfaces of infinite type and try to conclude a systematic way to do that.

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