Homology stability of mapping class groups

0. group homology

<u>Def</u>: G is a group, M is a $\mathbb{Z}[G]$-module. the i-th homology of G w/ coefficient M is

- (topologically) take any $K(G,1)$ space X,

$$H_i(G;M) := H_i(X;M)$$

- (algebraically) take any proj. resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$

$$H_i(G;M) := H_i(P_\bullet \otimes_{\mathbb{Z}[G]} M) \quad (= H_i(P_\bullet \otimes_G M))$$

<u>Rmk</u>: ① A canonical way to construct X:

classifying space of G: BG : 1 vertex

n-cells $\longleftrightarrow (g_1, \dots, g_n)$, $g_i \in G$

② Two def's are essentially the same: e.g. take $P_\bullet = C_\bullet(\tilde{X})$

1. homology stability

<u>Def</u>: A family of groups $G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \cdots$ satisfies <u>homological stability</u> if the induced maps $H_i(G_n) \to H_i(G_{n+1})$ are isomorphisms in a range $n \gg i$. $(\forall i)$

<u>Rmk</u>: ① $G_\infty := \varinjlim G_n$ is called the <u>stable group</u>

$H_i(G_\infty)$ is called the i-th <u>stable</u> ~~ith~~ homology

Actually, $\varinjlim H_i(G_n) \longrightarrow H_i(G_\infty)$ is an isomorphism.

(reason: from construction of BG, we can see inclusion of spaces

$BG_1 \hookrightarrow BG_2 \hookrightarrow \cdots \hookrightarrow BG_n \hookrightarrow \cdots$ and $BG_\infty = \bigcup_n BG_n$

from Hatcher Prop. 3.33, $\varinjlim_n H_i(BG_n) \longrightarrow H_i(BG_\infty)$ is an isom.)

(1.5) abused w/ no inclusion!

② Why useful?

- Only need to compute homology for small n to get all
- Properties of G_n & invariants $\longleftrightarrow$ Properties of G_∞ & invariants

<u>examples</u>: symmetric groups $S_1 \hookrightarrow S_2 \hookrightarrow \cdots$; linear gps $GL_1(\mathbb{Z}) \hookrightarrow GL_2(\mathbb{Z}) \hookrightarrow \cdots$ are homolog. stable.

2. General ~~???~~ Method (by Quillen in 1970s)

<u>Step 1</u>: Find CW complexes ~~of~~ X_n with G_n action s.t.

① X_n are highly connected

② Stabilizers of simplices are G_m, $m < n$

<u>Step 2</u>: Apply the equivariant homology spectral seq

Def: G is a group. X is a CW-cpx w/ G action, $C_\cdot(X)$ cellular chain cpx. the equivariant homology of (G, X) is

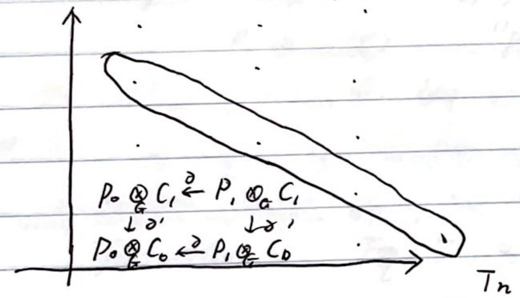
$$H_i^G(X;\mathbb{Z}) := H_i(G; C_\cdot(X))$$

$$(P_\cdot \to \mathbb{Z} \text{ proj. resol}/_{\mathbb{Z}[G]}) = H_i(P_\cdot \otimes_G C_\cdot(X))$$

Let $T_\cdot = P_\cdot \otimes_G C_\cdot(X)$, $T_n = \bigoplus_{s+t=n} P_s \otimes_G C_t(X)$

w/ differential $(P_\cdot, \partial)$ $(C_\cdot(X), \partial')$

$(T_\cdot, d)$ $\quad d = \partial + (-1)^p \partial'$



Observe: ① T_n has two natural filtrations $(F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_n = T_n)$

Filtration 1: $F_p(T_n) = \bigoplus_{i \le p} P_i \otimes_G C_{n-i}(X)$

Filtration 2: $F_q(T_n) = \bigoplus_{j \le q} P_{n-j} \otimes_G C_j(X)$

② Filtrations give us spectral seq's.

Def. A homological spectral sequence over a ring R contains

r-th page: bigraded R-modules $E^r_{p,q}$ $(p, q \in \mathbb{N})$

with differential $d^r: E^r_{p,q} \to E^r_{p-r,\, q+r-1}$ of degree $(-r, r-1)$

s.t. • $d^r \circ d^r = 0$

• $E^{r+1}_{p,q} = \ker(E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,\, q+r-1}) / \operatorname{Im}(E^r_{p+r,\, q-r+1} \xrightarrow{d^r} E^r_{p,q})$

e.g. 0-page $(0,-1)$ $\quad E_{0,1}\ E_{1,1}$ / $E_{0,0}\ E_{1,0}$ $\qquad$ 1-page $(-1, 0)$

Rmk: ① For each (p,q), $d^r|_{(p,q)} = 0$, $r > p+q+2$

the stabilized term is denoted as $E^\infty_{p,q}$, called ∞-page

② We say a spectral seq $E^r_{p,q} \Rightarrow M_{p+q}$ (R-mod) converges

if there's a filtration of M_n: $0 \hookrightarrow F_0(M_n) \hookrightarrow \cdots \hookrightarrow F_n(M_n)$

s.t. $E^\infty_{p,n-p} \cong F_p(M_n) / F_{p-1}(M_n)$

~~Cor:~~ Spectral Sequence 1 :

filtration 1 $F_p(T_n) = \bigoplus_{i\le p} P_i \otimes_G C_{n-i}(X)$

$\rightsquigarrow \quad E^0_{p,q} = F_p(T_{p+q}) / F_{p-1}(T_{p+q}) = P_p \otimes_G C_q(X)$

$d^0 : E^0_{p,q} \to E^0_{p,q-1} \qquad = (-1)^p \partial'$

$\quad P_p \otimes_G C_q(X) \qquad P_p \otimes_G C_{q-1}(X)$

$\Rightarrow E^1_{p,q} = P_p \otimes_G H_q(C_\cdot(X)) = P_p \otimes H_q(X)$

$d^1 : E^1_{p,q} \to E^1_{p-1,q} \qquad = \partial$

$\quad P_p \otimes H_q(X) \qquad P_{p-1} \otimes H_q(X)$

$\Rightarrow E^2_{p,q} = H_p(G; H_q(X))$

(☆1) $E^2_{p,q} = H_p(G; H_q(X)) \Rightarrow H_{p+q}(T_\cdot) = H^G_{p+q}(X)$

Spectral sequence 2 :

filtration 2 $F_q(T_n) = \bigoplus_{j\le q} P_{n-j} \otimes_G C_j(X)$

$E^0_{p,q} = F_p(T_{p+q}) / F_{p-1}(T_{p+q}) = P_q \otimes_G C_p(X)$

$d^0 : E^0_{p,q} \to E^0_{p,q-1} \qquad = \partial$

$\quad P_q \otimes_G C_p(X) \qquad P_{q-1} \otimes_G C_p(X)$

$\Rightarrow E^1_{p,q} = H_q(P_\cdot \otimes_G C_p(X)) = H_q(G; C_p(X))$

$* \; C_p(X) = \bigoplus_{p\text{-cells of } X} \mathbb{Z} = \bigoplus_{\substack{\sigma\in G\text{-rep of } p\text{-cells} \\ \Sigma_p}} (G \otimes_{Stab(\sigma)} \mathbb{Z}) = \bigoplus_{\sigma\in\Sigma_p} Ind^G_{Stab(\sigma)} \mathbb{Z}$

Shapiro's Lem $H_q(G; \bigoplus_{\sigma\in\Sigma_p} Ind^G_{Stab(\sigma)} \mathbb{Z})$

$\cong \bigoplus_{\sigma\in\Sigma_p} (H_q(G; Ind^G_{Stab(\sigma)} \mathbb{Z})$

$\cong \bigoplus_{\sigma\in\Sigma_p} H_q(Stab(\sigma); \mathbb{Z})$

(☆2) $E^1_{p,q} = \bigoplus_{\sigma\in\Sigma_p} H_q(Stab(\sigma)) \Rightarrow H^G_{p+q}(X)$

Rmk : ~~(☆1) gives ... implies~~

X highly connected $\Rightarrow$ (☆1) many 0's

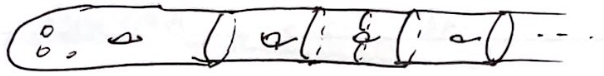
$Stab(\sigma)$ being G_m, $m<n$ $\Rightarrow$ (☆2) gives us $H_i(G_m) \to H_i(G_n)$

3. Homology stab. for mapping class groups

$\Sigma_{g,b}$: oriented surface of genus g w/ b boundary cpts

$Mod(\Sigma_{g,n}) =$ ~~{orientati~~ $Diff^+(\Sigma_{g,n}, \partial\Sigma_{g,n})$ / isotopy fixing boundary

Natural Inclusions of MCG's w/ boundary:

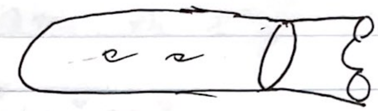


($r \geq 1$) $\mathrm{Mod}(\Sigma_{g,r}) \hookrightarrow \mathrm{Mod}(\Sigma_{g+1,r}) \hookrightarrow \mathrm{Mod}(\Sigma_{g+2,r}) \hookrightarrow \cdots$

extend by identity

Decompose into two steps: $\beta_g \circ \alpha_g$

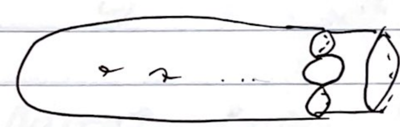
① α_g :



$b: +1$

$r \geq 1$ $\quad \mathrm{Mod}(\Sigma_{g,r}) \to \mathrm{Mod}(\Sigma_{g,r+1})$

② β_g



genus: $+1$
$b: -1$

$r \geq 1$ $\quad \mathrm{Mod}(\Sigma_{g,r+1}) \to \mathrm{Mod}(\Sigma_{g+1,r})$

③ δ_g

$\mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_{g,0})$

Thm (Harer Stability theorem)

① $g \geq 0$, $r \geq 1$, $H_n(\alpha_g)$ is iso for $g \geq \frac{3}{2}n$

② $g \geq 0$, $r \geq 1$, $H_n(\beta_g)$ is iso for $g \geq \frac{3}{2}n + 1$

③ $H_n(\delta_g)$ is iso for $g \geq \frac{3}{2}n$

Rmk: ① These shows the n-th homology of $\mathrm{Mod}(\Sigma_{g,r})$ is indep of g & $r (\geq 0)$, whenever $g \geq \frac{3}{2}n + 1$

② The ranges of stability are given by:

Harer: $3n$ $\quad$ Ivanov: $2n$ $\quad$ Boldsen, Randal-Williams $\frac{3}{2}n$

Best possible when $g \equiv 2 \pmod 3$, at most 1 off o/w.

Proof: Use $\mathrm{Mod}(\Sigma_{g,r}) \curvearrowright$ special curve cpx (lecture 2)

Q. What's the stable homology?

Madsen-Weiss Thm (lecture 3)

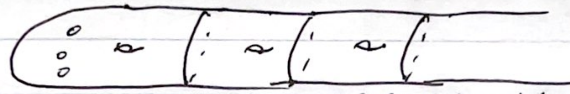
Characteristic classes of surface bdls (lecture 4)

Homology Stability of Mapping Class Groups II

$\Sigma_{g,b}$: oriented surface of genus g, with b boundary components

Mapping class group of $\Sigma_{g,b}$: $Mod(\Sigma_{g,b}) = Diff^+(\Sigma_{g,b}, \partial\Sigma_{g,b})$

/ isotopy fixing ∂ pointwise

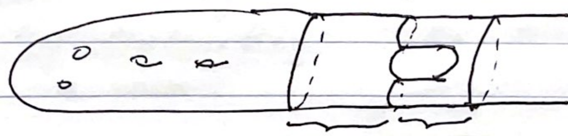


(extend by id)

$b \geq 1$, $Mod(\Sigma_{g,b}) \underset{\gamma_g}{\hookrightarrow} Mod(\Sigma_{g+1,b}) \underset{\gamma_{g+1}}{\hookrightarrow} Mod(\Sigma_{g+2,b}) \hookrightarrow \cdots$

is homologically stable, i.e. $H_i(Mod_{g,b}) \to H_i(Mod_{g+1,b})$ is o for $g >> i$.

Separate into two steps:



α_g β_g

α_g : $Mod(\Sigma_{g,b}) \hookrightarrow Mod(\Sigma_{g,b+1})$ $\quad b \geq 1$

β_g : $Mod(\Sigma_{g,b+1}) \to Mod(\Sigma_{g+1,b})$

$\gamma_g = \beta_g \circ \alpha_g$

<u>Thm1</u> (Hatcher & Vogtmann) The induced maps on homology

(i) $H_i(\alpha_g)$: $H_i(Mod_{g,b}) \to Mod(\Sigma_{g,b+1})$

(ii) $H_i(\beta_g)$: $H_i(Mod_{g,b+1}) \to Mod(\Sigma_{g+1,b})$

[Sharper: $g \geq \frac{3}{2}i$, $g \geq \frac{3}{2}+1$

are isomorphisms for $g \geq 2i+2$.

<u>Rmk</u> : (1) (i)&(ii) shows $H_i(\gamma_g = \beta_g \circ \alpha_g)$ is isom. for $g \geq 2i+2$

(2) Shows more! + (i) shows $H_i(Mod_{g,b})$ is indep. of g <u>and</u> <u>$b \geq 1$</u> for $g \geq 2i+2$

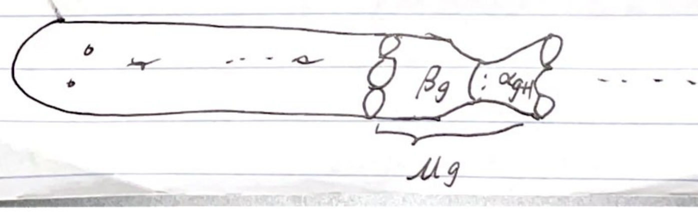
<u>Thm2</u> (Hatcher & Vogtmann, or Nathalie)

δ_g : $Mod(\Sigma_{g,1}) \to Mod(\Sigma_{g,0})$

$H_i(\delta_g)$: $H_i(Mod_{g,1}) \to H_i(Mod_{g,0})$ is an isom. for $g \geq 2i+3$

<u>Goal Today</u>: A proof sketch for Thm 1.

<u>claim</u>: Suffices to prove $H_i(\mu_g := \alpha_{g+1} \circ \beta_g)$ isom for $g \geq 2i+2$, surj for $g = 2i+1$



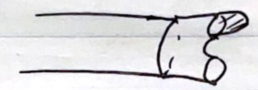
μ_g

$b+1 \geq 2$ $\quad \mathrm{Mod}_{g,b+1} \xrightarrow{\mu_g} \mathrm{Mod}_{g+1,b+1} \xrightarrow{\mu_{g+1}} \mathrm{Mod}_{g+2,b+1}, \cdots$

Pf of Claim:

$H_i(\mu_g = \alpha_{g+1} \circ \beta_g)$ surj $g \geq 2i+1 \Rightarrow H_i(\alpha_g)$ surj. $g \geq 2i+2$

Notice: $H_i(\alpha_g)$ is always injective:



capping one bdry cpt is left inverse.

$\Rightarrow$ $H_i(\alpha_g)$ isom for $g \geq 2i+2$

$\Rightarrow$ $H_i(\beta_g)$ isom for $g \geq 2i+2$ ✓

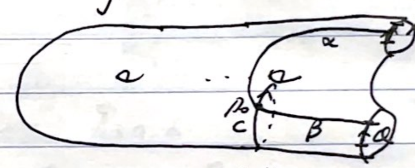
Recall Quillen's general method for $G_1 \hookrightarrow G_2 \hookrightarrow \cdots$:

find CW cpx $X_n \circlearrowleft G_n$

Now: $G_g = \mathrm{Mod}_{g,b}$ $(b \geq 2)$

Key construction: $X_g = DTC^m(\Sigma_{g,b}, P, Q)$

a vertex:

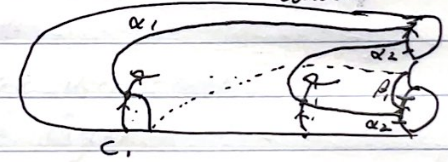


$[(C, p_0, \alpha, \beta)]$ isotopy class

C: a double tethered curve, nonsep.

α, β: disjoint exept p_0

an edge:



$[[C_1, p_0, \alpha, \beta_1], [C_2, p_2, \alpha_2, \beta_2]]$

disjoint. $\{C_1, C_2\}$ coconnect

$\Rightarrow$ dou. tethered curves coconnect

orders of tethers on P = orders on tethers on Q

...

k simplex: an isotopy class of a system of

$(k+1)$ disjoint ~~disjoint~~ double tethered curves

s.t. the complement is connected

and orders on P, Q are matching.

Step 1:

Thm: $DTC^m(\Sigma_{g,b}, P, Q)$ is $\frac{g-3}{2}$ connected; is $(g-1)$-dim'l.

Pf: $C(\Sigma_{g,b}) \dashrightarrow C^\circ(\Sigma_{g,b}) \dashrightarrow TC(\Sigma_{g,b}, P) \dashrightarrow DTC(\Sigma_{g,b}, P, Q)$

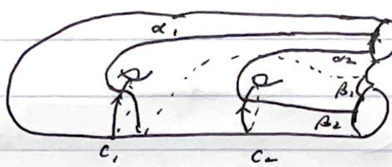
$(2g-3)$-connected $\quad$ coconnected $\quad$ $\geq \left(\frac{g-3}{2}\right)$

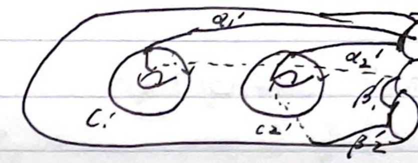
$(g-2)$-connected

link argument & surgery argument $\quad$ $DTC^m(\Sigma_{g,b}, P, Q)$

Step 2: $\underset{``G_g}{Mod_{g,b}} \curvearrowright \underset{``X_g}{DTC^m(\Sigma_{g,b}, P, Q)}$ "nicely"

① transitive on k-simplices:





$(c_1, \alpha_1, \beta_1) \quad (c_2, \alpha_2, \beta_2) \qquad (c_1', \alpha_1', \beta_1') \quad (c_2', \alpha_2', \beta_2')$

cut $(c_1, c_2) \to (c_1', c_2')$ first

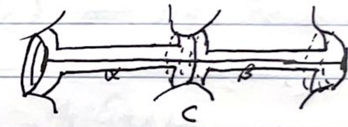
orders on P, Q matches $\Rightarrow$ $\alpha_i \mapsto \alpha_i'$

$\beta_i \mapsto \beta_i'$

② Stabilizers of a k-simplex $\cong G_{g-k-1}$

· v: a vertex

$Nb(v) \cong \Sigma_{0,4}$



$\Sigma_{g,b} = \Sigma_{g,b} \setminus Nb(v) \xrightarrow{\mathcal{U}_{g-1}} \Sigma_{g,b}$

$Stab(v) \cong Mod(\Sigma_{g-1,b}) = G_{g-1}$

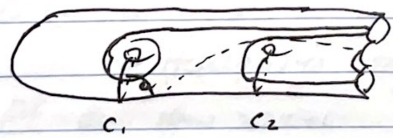
· σ: a k-simplex

do above for $(k+1)$-times

$Stab(\sigma) = G_{g-k-1}$

③ $\exists f \in G_g$, f commutes w/ $Stab(e)$, ~~f permutes v_1, v_2~~ $f(v_1) = v_2$

an edge $v_1 \overset{e}{—} v_2$



$e = [\underbrace{(c_1, \alpha_1, \beta_1)}_{v_1}, \underbrace{(c_2, \alpha_2, \beta_2)}_{v_2}]$

$Nb(e) \cong \Sigma_{1,2}$

Since $Mod(\Sigma_{1,2}) \overset{\text{transitive}}{\curvearrowright} DTC^m(\Sigma_{1,2}, P, Q)$

$\exists f$ ~~permutes~~ maps v_1 to v_2, supported on $Nb(e)$

Notice $Stab(e)$ fixes e pointwise $\Rightarrow f$ commutes w/ $Stab(e)$

Step 3: Two spectral sequences converge to "reduced" equivariant homology

$G_g \curvearrowright X_g$

$P_\cdot \to \mathbb{Z}$ proj. resolution over $\mathbb{Z}[G_g]$

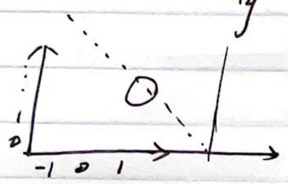
$\tilde{C}_\cdot(X_g): \cdots \to C_0(X_g) \to C_0(X_g) \to \underset{\overset{\shortparallel}{\mathbb{Z}}}{C_{-1}(X_g)} \to 0$

$\tilde{H}^{G_g}_*(X_g) = \tilde{H}_*(\cancel{P_\cdot \otimes_{G_g}}\ \tilde{C}_\cdot(X_g) \otimes_{G_g} P_\cdot)$

① Spectral sequence 1

$$E^2_{p,q} = H_q(G_g; \tilde{H}_p(\mathring{X}_g)) \Rightarrow \tilde{H}^{G_g}_{p+q}(X_g; \mathbb{Z})$$

$p \geq -1,\ q \geq 0$



X_g $\frac{g-3}{2}$ connected $\Rightarrow \tilde{H}_p(X_g) = 0,\ -1 \leq p \leq \frac{g-3}{2}$

$\Rightarrow E^\infty_{p,q} = 0,\ p+q \leq \frac{g-3}{2}$

$\Rightarrow \tilde{H}^{G_g}_{p+q}(X_g; \mathbb{Z}) = 0,\ p+q \leq \frac{g-3}{2}$

② Spectral sequence 2

$$E^1_{p,q} = \bigoplus_{\sigma \in p\text{-cell rep.}} H_q(\mathrm{Stab}(\sigma)) \Rightarrow \tilde{H}^{G_g}_{p+q}(X_g; \mathbb{Z})$$

$\|$ transitive

$$H_q(\mathrm{Stab}(\Delta^p)) \overset{\text{stab}}{=\!=} H_q(G_{g-p-1})$$

$H_i(G_{g-1}) = d^1$

i: $H_i(G_g) \leftarrow H_i(G_{g-1}) \leftarrow H_i(G_{g-2})$

d^2 surj. by induction

$H_{i-1}(G_g) \leftarrow H_i(G_{g-1}) \leftarrow H_i(G_{g-2}) \leftarrow H_i(G_{g-3})$ $\quad g-3 \geq 2(i-1)+1$

$q=1$

$q=0$: $H_0(G_g) \leftarrow H_0(G_{g-1}) \leftarrow H_0(G_{g-2}) \leftarrow$

$p = -1 \quad 0 \quad 1 \quad 2 \ \ldots$

Goal: ① $\underbrace{H_i(G_{g-1})}_{d^1}$ is surj. for $(g-1) \geq 2i+1$

② $\underbrace{H_i(G_{g-1})}_{d^1}$ is inj. for $(g-1) \geq 2i+2$

Pf: ① $(g-1) \geq 2i+1 \Rightarrow \#i \leq \frac{g-2}{2} \leq \frac{g-3}{2}$

$\Rightarrow E^\infty_{-1,i} = 0$

Suffices to prove $d^r: H_i(G_g) = E^r_{-1,i} \leftarrow E^r_{-1+r,\, i-r+1} = 0$ ($d^r_{r\geq 2}$)

but! $E^2_{-1+r,\, i+r+1} = 0$ by induction on i

② similar to ①, can find d^r to $H_i(G_{g-1}) = 0$ ($r \geq 2$)

but: need to verify $E^1_{0,i} \leftarrow E^1_{1,i}$ is 0-map

$E^1_{0,i} = H_i(G_{g-1}) = H_i(\mathrm{Stab}(v_0))$, $E^1_{1,i} = H_i(G_{g-2}) = H_i(\mathrm{Stab}(e))$

Use property 3:

$$\begin{tikzcd} & H_i(\mathrm{Stab}(v_1)) \arrow[d, "f_*"] \arrow[dr] & \\ H_i(\mathrm{Stab}(e)) \arrow[ur] \arrow[r] \arrow[urr, bend left, "\partial_1"] \arrow[drr, bend right, "\partial_2"'] & H_i(\mathrm{Stab}(v_2)) \arrow[r] & H_i(\mathrm{Stab}(v_0)) \end{tikzcd}$$

$= \partial_1 - \partial_2 = 0$!

Q : What are the stable homology groups of $Mod_{g,b}$?

A' : Rational stable cohomology groups are

$$H^*(\varinjlim_g Mod_{g,b}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \cdots] \quad |\kappa_i| = 2i$$

κ_i : Mumford-Morita-Miller classes ... next lecture

(Mumford Conjecture, proved by Madsen-Weiss)

Thm : (Madsen-Weiss Theorem)

$$H_*(\varinjlim_g Mod_{g,b}) \cong H_*(\Omega_0^\infty AG^+_{\infty,2})$$

$$\cong H_*(\Omega_0^\infty MTSO(2)) := H_*(\varinjlim_n \Omega_0^n Th(\gamma_n^\perp))$$

$$\cong H_*(\Omega_0^\infty \Psi)$$

Using Harer Stability, restate: $H_i(Mod_{g,1}) \to H_i(\Omega_0^\infty MTSO(2))$ is an isom for $g \geq \frac{3}{2}i + 1$

Notations :

① subscript 0 : the connected component of the basepoint

② $\Omega^\infty AG^+_{\infty,2} = \varinjlim_n \Omega^n AG^+_{n,2}$

$AG^+_{n,2} = \{ P :$ oriented flat 2-plane in $\mathbb{R}^n \}$'s onept compactification (ϕ)

$AG^+_{n,2} \to \Omega AG^+_{n+1,2}$ translate along $(n+1)$-st coordinates

$\rightsquigarrow \Omega^n AG^+_{n,2} \to \Omega^{n+1} AG^+_{n+1,2}$

③ $\gamma_n \downarrow Gr^+_{n,2}$ = canonical bdl over oriented Grassmannian $Gr^+_{n,2} = \{ W \ni 0 :$ oriented 2-plane in $\mathbb{R}^n \}$

$\{ (W, x) \in Gr^+_{n,2} \times \mathbb{R}^n \mid x \in W \}$

$\gamma_n^\perp \downarrow Gr^+_{n,2}$ orthogonal bdl $\gamma_n^\perp = \{ (W, v) \in Gr^+_{n,2} \times \mathbb{R}^n \mid v \perp W \}$

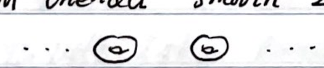
$Th(\gamma_n^\perp)$ Thom space

Prop : $Th(\gamma_n^\perp) \to AG^+_{n,2}$ is a weak h.e. (i.e. π_i isom's)

$(W, v) \mapsto W + v$

basept $\mapsto \phi$

④ $\Psi(\mathbb{R}^n) = \{ M \subseteq \mathbb{R}^n : M$ oriented smooth 2-mfd, topologically closed (≠ closed) without boundary

e.g. 

$\Psi(\mathbb{R}^n) \to \Omega \Psi(\mathbb{R}^{n+1})$ Translate along $(n+1)$-st coor

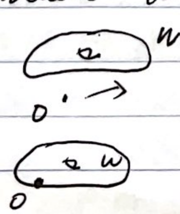
$W \mapsto (t \mapsto W \times \{t\})$

$\infty \mapsto \phi$

$\rightsquigarrow \Omega^n \Psi(\mathbb{R}^n) \to \Omega^{n+1} \Psi(\mathbb{R}^{n+1})$

Prop: $\Psi(\mathbb{R}^n)$ is homotopy equivalent to its subspace $AG^+_{n,2}$.

Pf sketch : rough idea :



stretch to $\infty \rightsquigarrow \phi$

rescaling to $\infty \rightsquigarrow$ tangent plane at 0

Problem: : Not continuous if we perturb M a little bit

solution: Modify the rescaling operation to a tubular nbhd W of M :

tangential direction: rescaling from 1 to ∞

normal direction: $\begin{cases} \text{~~if o to~~ if } 0 \text{ near } M, & \text{near } 1 \\ \text{if } 0 \text{ near } \partial W, & \text{near } \infty \end{cases}$

Thus $\psi(\mathbb{R}^n)$ def. retract to $AGr^+_{n,2}$.

<u>Goal Today</u>: A proof sketch for $H_i(Mod_{g,0}) \to H_i(\Omega_0^\infty \psi)$ is an isom. for $g \geq \frac{3}{2} i + 1$

<u>Step 1</u>: Find a nice $K(Mod_{g,1}, 1)$

$B_n := \{ M \in \psi(\mathbb{R}^n),\ M \subseteq (0,1)^n \}$ $(n \geq 5)$

$= \coprod_M Emb(M, (0,1)^n) / Diff^+(M)$

diffeo class: $= \coprod_M BDiff^+(M)$

$BDiff^+(\Sigma_{g,1}) \hookrightarrow B_n$

Observe: $BDiff^+(\Sigma_{g,0})$ is a $K(Mod_{g,1}, 1)$ space, $g \geq 2$

pf: $Diff^+(\Sigma_{g,0}) \to EDiff^+(\Sigma_{g,0})$

$\downarrow$

$BDiff^+(\Sigma_{g,0})$

LES on homotopy groups $\Rightarrow \pi_i(BDiff^+(\Sigma_{g,0})) \cong \pi_{i-1}(Diff^+(\Sigma_{g,0}))$

<u>The Earle-Eells Theorem</u>: S is a compact connected surface, then the components of $Diff^+(S, \partial S)$ is contractible except when $S = S^2, T^2, \mathbb{RP}^2$, klein bottle.

Thus $\pi_{\geq 1}(Diff^+(\Sigma_g)) = 0$

$\Rightarrow \pi_1(BDiff^+(\Sigma_g)) = \pi_0 Diff^+ \Sigma_g = Mod_g$

$\pi_{\geq 2}(BDiff^+(\Sigma_g)) = 0$ $\square$

<u>Step 2</u>: $BDiff^+(\Sigma_g) \hookrightarrow \underset{= \psi(n,0)}{B_n} \xrightarrow{\alpha_0} \Omega\psi(n,1) \xrightarrow{\Omega\alpha_1} \Omega^2\psi(n,2) \xrightarrow{\Omega^2\alpha_2} \cdots \to \Omega^n \psi(n,n) = \Omega^n \psi(\mathbb{R}^n)$

<u>Def</u>: $\psi(n,k) = \{ M \in \psi(\mathbb{R}^n),\ M \subseteq \mathbb{R}^k \times (0,1)^{n-k} \}$

$B_n = \psi(n,0)$

$\alpha_k : \psi(n,k) \to \Omega\psi(n,k+1)$

$M \mapsto (t \mapsto M + t e^{k+1})$

<u>Step 2.1</u>: $k \geq 1$, $\alpha_k : \psi(n,k) \to \Omega\psi(n,k+1)$ is a w.h.e.

<u>Step 2.2</u>: $k = 0$, let $n \to \infty$, $BDiff^+(\Sigma_g) \to \Omega\psi(\infty, 1)$ induces isom on $H_i(-)$ for $i < \frac{2(g-1)}{3}$.

Proof of Step 2.1

Def: A space M is a topological monoid if there's a continuous map $\mu: M\times M \to M$ which is associative, and has an identity e.

Def: The classifying space of a top'l monoid M is:

$BM := \coprod_p \Delta^p \times M^p \,/\sim$

(also called ~~geometrization~~ realization of the nerve of M)

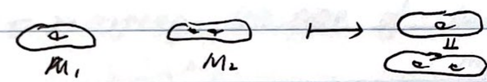
D^m

natural map $M \longrightarrow \Omega BM$

$m \longmapsto (S^1 \mapsto \Delta^1 \times \{m\})$

Thm: $M \to \Omega BM$ is a w.h.e. $\Leftrightarrow$ $\pi_0 M$ is a group

Observe: $n-1 \geq k \geq 0$ $\quad \psi(n,k) \times \psi(n,k) \longrightarrow \psi(n,k)$

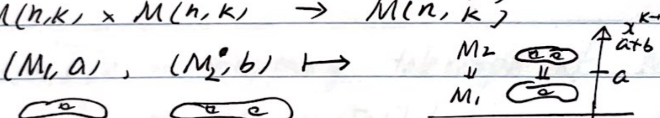


homotopy-associative, has homotopy-id ϕ

Def: $M(n,k) := \{(M,a) \in \psi(\mathbb{R}^n) \times [0,\infty) \mid M \subseteq \mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1}\}$

$M(n,k) \times M(n,k) \to M(n,k)$

$(M_1, a), \ (M_2, b) \mapsto$



Associative ✓, identity $e = \phi$ ✓

Note: $\psi(n,k) \hookrightarrow M(n,k)$ is a h.e.

$M \longmapsto (M, 1)$

Goal: $k \geq 1$ $\quad \psi(n,k) \longrightarrow \Omega\, \psi(n,k+1)$ h.e.

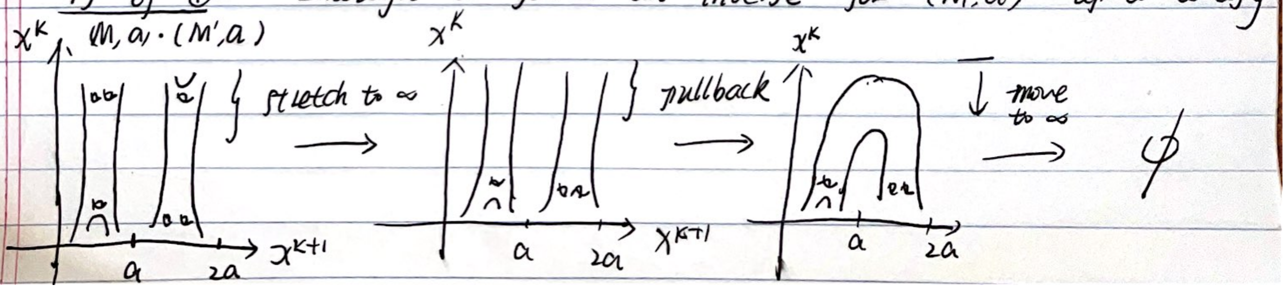
$$M(n,k) \xrightarrow[(\star_1)]{\simeq} \Omega BM(n,k) \xrightarrow[(\star_2)]{\simeq} \Omega\, \psi_0(n,k+1)$$

To $\star_1, \star_2$:

Thm: ① $\pi_0(M(n,k))$ is a group, if $k \geq 1$

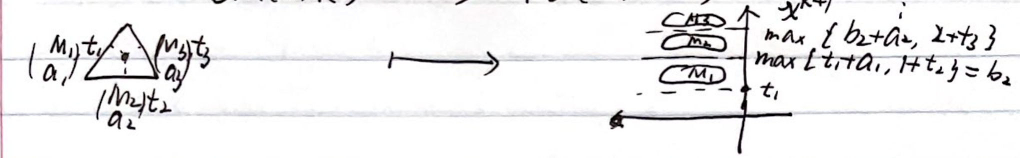
② $BM(n,k) \to \psi_0(n,k+1)$ is a w.h.e., $k \geq 1$

Pf of ①: Enough to find an inverse for (M,a) up to isotopy



Idea for ②:

$$BM(n,k) \longrightarrow \Psi_0(n, k+1)$$



to prove w.h.e., inj. & surj. on π_i.

Proof for Step 2.2:

Problem: $\pi_0(M(n,0)) \cong \mathbb{Z}_{\geq 0}$ (genus) not a group!

Thm (Group Completion Thm) If $\pi_0 M = \mathbb{Z}_{\geq 0}$ for a top'l monoid M, and M is homotopy commutative, then

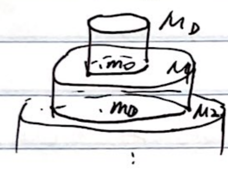
$$H_i(TM) \cong H_i(\Omega BM), \quad H_i(M_\infty) \cong H_i(\Omega_0 BM)$$

restricting to one component

where M_∞ = mapping telescope of $M_0 \xrightarrow{\cdot m_0} M_1 \xrightarrow{\cdot m_0} M_2 \xrightarrow{\cdot m_0} \cdots$

(m_0 generates π_0)

=



TM = mapping telescope of $M \xrightarrow{\cdot m_0} M \xrightarrow{\cdot m_0} M \to \cdots$

$= M \times \mathbb{N} \times [0,1] / (m, n, 1) \sim (m_1 \cdot m_0, n+1, 0)$

= mapping telescope of

$$\begin{array}{ccc} M_0 & M_0 & M_0 \\ M_1 & M_1 & M_1 \\ M_2 & M_2 & M_2 \\ \vdots & & \end{array}$$

$= \mathbb{Z} \times M_\infty$

Nb: $H_i(M_\infty) = \varinjlim_n H_i(M_n)$ if $M_n \to M_{n+1}$ is a cofibration.